An abstract existence result for generalised Hughes' model

From conjoint works with B. Andreianov

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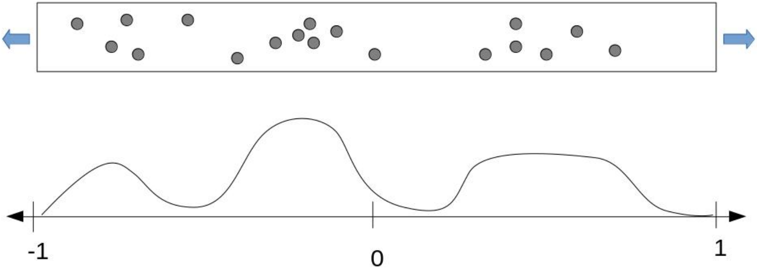
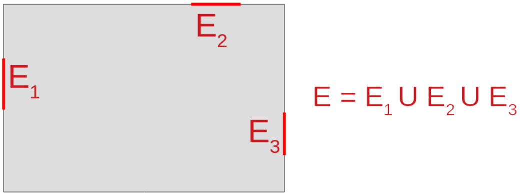
January 18, 2023

Outline

We want to study the macroscopic quantity that is the crowd density:



- $\rho \in [0, 1]$
- $\rho(t, x)$, $t \in [0, T]$, $x \in \mathbb{R}$



The domain is a room D. At $t \leq 0$, the pedestrians are distributed as an initial density ρ_0. Starting at $t = 0$, the pedestrians want to leave D through the exits E.

We consider the following eikonal equation for the potential:

$$\begin{cases} |\nabla\phi| = \dfrac{1}{g(\rho)v(\rho)} & \\ \phi(t,x) = 0 & \text{if } x \in E \\ \nabla\phi(t,x)\cdot\nu(x) = 0 & \text{if } x \in \partial D \backslash E. \end{cases}$$

Here ν is the normal unit vector. The decreasing g function takes into account the disconfort for pedestrian to stay in high density region.

The Lighthill - Whitham and Richards (LWR) model for traffic flow
:

$$\begin{cases} \rho_t + [\rho v(\rho)]_x = 0 \\ \rho(0,\cdot) = \rho_0(\cdot), \end{cases}$$

with $v(\rho) = 1 - \rho$.

We set $f(\rho) = \rho(1-\rho)$, f is concave.

• M. J. Lighthill and G. B. Whitham, On kinematic waves. ii. a theory of traffic flow on long crowded roads, (1955).
• P. I. Richards, Shock waves on the highway, (1956).
• M. Di Francesco and M. Rosini, Rigorous derivation of nonlinear scalar conservation laws from follow-the-leader type models via many particle limit, 2015.

Consequently, the density of pedestrian will move, satisfying a LWR one-dimensional conservation law equation, in the direction of the descending gradient i.e.

$$\begin{cases} \rho_t + \mathrm{div}\left[\dfrac{-\nabla\phi}{|\nabla\phi|}\rho v(\rho)\right] = 0 \\ \rho(0,x) = \rho_0(x) \\ \rho(t, x \in E) = 0 \end{cases}$$

This corresponds to a discontinuous flux conservation law.
This boundary condition is equivalent to open-end exit condition.

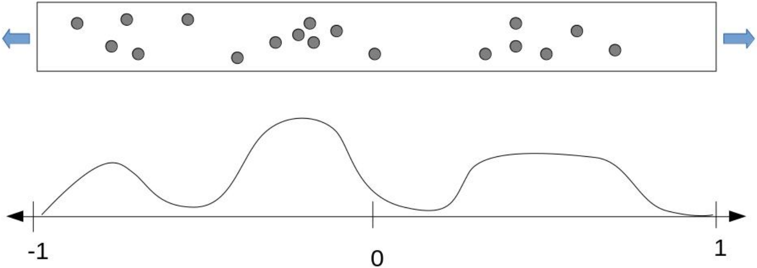
The original multi-D Hughes' model:

$$\begin{cases} \rho_t + \operatorname{div}\left[\dfrac{-\nabla\phi}{|\nabla\phi|}\rho v(\rho)\right] = 0 & (2a) \\ |\nabla\phi| = \dfrac{1}{g(\rho)v(\rho)} & (2b) \\ \phi(t, x \in E) = 0 & (2c) \\ \nabla\phi(t, x \in \partial D \backslash E) \cdot \nu(x) = 0. & (2d) \end{cases}$$

Introduced in :
R. L. Hughes, *A continuum theory for the flow of pedestrians*, Transportation Research Part B-methodological, 36 (2002), pp. 507–535.

In one dimension, think of a corridor:

$$\begin{cases} \rho_t + [\mathrm{sign}(-\partial_x \phi)\rho v(\rho)]_x = 0 \\ |\partial_x \phi| = \dfrac{1}{g(\rho)v(\rho)} \\ \phi(t, x = \pm 1) = 0. \end{cases}$$



We can try to reformulate the problem in terms of control trajectories:

$$\begin{cases} \dot{y}_x(t) = \alpha \in \{\pm 1\} \\ y_x(0) = x. \end{cases}$$

With

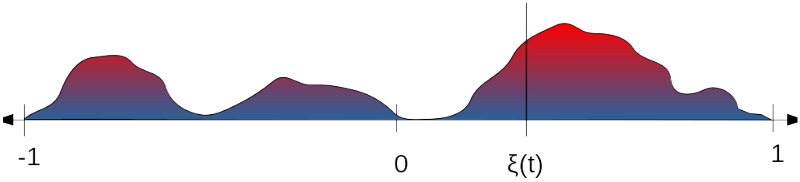
$$T := \inf \left\{ t \in \mathbb{R}^+ \text{ s.t. } y_x(t) \in \{\pm 1\} \right\},$$

we write the following value function :

$$\phi(t,x) := \inf_{\alpha} \left\{ \int_0^T c(\rho(t, y_x(s))) \, \mathrm{d}s \right\}$$

where $c(\rho) = \frac{1}{g(\rho)v(\rho)}$.

$$\phi(t,x) := \min \left\{ \int_x^1 c(\rho(t,y)) \, \mathrm{d}y, \int_{-1}^x c(\rho(t,y)) \, \mathrm{d}y \right\}$$



The turning curve approach

$$\begin{cases} \rho_t + [\mathrm{sign}(x - \xi(t))\rho v(\rho)]_x = 0 & \text{(5a)} \\ \int_{-1}^{\xi(t)} c(\rho(t,x))\,\mathrm{d}x = \int_{\xi(t)}^{1} c(\rho(t,x))\,\mathrm{d}x, & \text{(5b)} \end{cases}$$

with c called the cost function.

If $c(\rho) = \frac{1}{g(\rho)v(\rho)}$ the Hughes' model (2) and (5) are equivalent.

The proof can be found in:
N. El-Khatib, P. Goatin, and M. D. Rosini, *On entropy weak solutions of hughes model for pedestrian motion*, Zeitschrift für angewandte Mathematik und Physik, 64 (2013), pp. 223–251.

We want to solve the one-dimensionnal problem:

$$\begin{cases} \rho_t + [\operatorname{sign}(x-\xi(t))\rho v(\rho)]_x = 0 \\ \displaystyle\int_{-1}^{\xi(t)} c(\rho(t,x))\,\mathrm{d}x = \int_{\xi(t)}^{1} c(\rho(t,x))\,\mathrm{d}x. \end{cases}$$

with a general c satisfying:

$$\begin{cases} c \in \mathcal{C}^0([0,1]), \\ \forall \rho \in [0,1], c(\rho) \geq 1, \\ c \text{ is increasing on } [0,1]. \end{cases}$$

- Well-posedness for the regularized Hughes' model:
  M. Di Francesco, P. A. Markowich, J.-F. Pietschmann, and M.-T. Wolfram, *On the hughes' model for pedestrian flow: The one-dimensional case*
- Well-posedness for the discontinuous-flux conservation law:
  N. El-Khatib, P. Goatin, and M. D. Rosini, *On entropy weak solutions of hughes model for pedestrian motion*
- Existence result if $\rho_{|\{x=\xi(t)\}} = 0$:
  D. Amadori, P. Goatin, and M. D. Rosini, *Existence results for hughes' model for pedestrian flows*
- Existence result for the affine cost case:
  B. Andreianov, M. D. Rosini, and G. Stivaletta, *On existence, stability and many-particle approximation of solutions of 1D Hughes model with linear costs.*

Coming soon : a chapter in Crowd dynamics volume 4 edited by Gibelli and Bellomo,

The mathematical theory of Hughes' model: a survey of results
Compiling the works of :
D. Amadori, B. Andreianov, M. Di Francesco, S. Fagioli, T. Girard, P. Goatin, P. Markowich, J.-F. Pietschmann, M.D. Rosini, G. Russo, G. Stivaletta and M.T. Wolfram

Fix $\xi \in W^{1,\infty}$.
We first discuss the notion of solution for :

$$\rho_t + [\text{sign}(x - \xi(t)) f(\rho)]_x = 0.$$

It is a **one-dimensional conservation law with discontinuous flux** (in $x = \xi(t)$).

Definition (Classical entropy solution)

$\rho \in L^\infty$ is an entropy solution to $\rho_t + f(\rho)_x = 0$ if

- ρ is a weak solution i.e for all $\phi \in \mathcal{C}_c^\infty$, $\phi \geq 0$:

$$\iint_{(0,T)\times\mathbb{R}} \rho\phi_t + f(\rho)\phi_x \,\mathrm{d}t\,\mathrm{d}x = 0$$

- ρ satisfies the entropy inequalities, for all ϕ, all $k \in \mathbb{R}$:

$$\iint_{(0,T)\times\mathbb{R}} |\rho - k|\phi_t + q_f(\rho, k)\phi_x \,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} |\rho_0(x) - k|\phi(0,x)\,\mathrm{d}x \geq 0$$

where $q_f(p_1, p_2) := \mathrm{sign}(p_1 - p_2)\,[f(p_1) - f(p_2)]$

Definition (N. El-Khatib, P. Goatin, and M. D. Rosini)

$\rho \in L^\infty$ is an entropy solution to $\rho_t + [\mathrm{sign}(x - \xi(t))\rho v(\rho)]_x = 0$ if ρ is a weak solution and satisfies the following entropy condition for all $k \in \mathbb{R}$:

$$\iint_{(0,T)\times\mathbb{R}} |\rho - k|\phi_t + q_F(\rho, k)\phi_x \,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} |\rho_0(x) - k|\phi(0, x)\,\mathrm{d}x + 2\int_0^T f(k)\phi(t, \xi(t))\,\mathrm{d}t \geq 0$$

where $q_F(p_1, p_2) := \mathrm{sign}(p_1 - p_2)\,[F(p_1) - F(p_2)]$
and $F(t, x, p) = \mathrm{sign}(x - \xi(t))p(1 - p)$

Definition (Entropy solution without interface condition)

$\rho \in L^\infty$ is an entropy solution to $\rho_t + [\mathrm{sign}(x - \xi(t)) f(\rho)]_x = 0$ if ρ is a weak solution and satisfies the following entropy condition for all $k \in \mathbb{R}$,

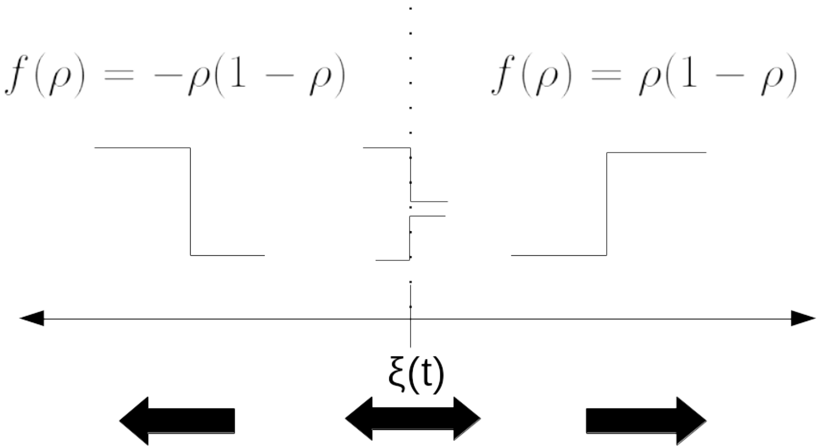
- for all ϕ supported in $\{x < \xi(t)\}$:

$$\iint_{\{x<\xi(t)\}} |\rho - k|\phi_t + q_{-f}(\rho, k)\phi_x \,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} |\rho_0(x) - k|\phi(0, x)\,\mathrm{d}x \geq 0$$

- for all ϕ supported in $\{x > \xi(t)\}$:

$$\iint_{\{x>\xi(t)\}} |\rho - k|\phi_t + q_f(\rho, k)\phi_x \,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} |\rho_0(x) - k|\phi(0, x)\,\mathrm{d}x \geq 0$$

A word about classical and non-classical shocks...



Theorem (Wellposedness of the discontinuous conservation law)

Let ρ_0 is $L^1(\mathbb{R})$. Let $\xi \in W^{1,\infty}((0,T))$. There exists a unique $\rho \in L^1((0,T)\times\mathbb{R})$ solution to $\rho_t + [\mathrm{sign}(x-\xi(t))\rho v(\rho)]_x = 0$ with initial datum ρ_0 in the sense of the previous definition.

Corollary

Fix $\rho_0 \in L^1(\mathbb{R})$. Then we can define the (non-linear) operator

$$\mathcal{S}_0 : \begin{cases} W^{1,\infty}((0,T)) \longrightarrow L^1((0,T)\times\mathbb{R}) \\ \qquad\qquad \xi \mapsto \rho. \end{cases}$$

This operator is well-defined and monovaluated.

Generalized Hughes' model :

$$\begin{cases} \rho_t + [\text{sign}(x - \xi(t)) f(\rho)]_x = 0 \\ \rho(0, x) = \rho_0(x) \\ \xi = \mathcal{I}(\rho) \end{cases}.$$

In particular, the turning curve model is a generalised Hughes' model with $\mathcal{I}$ that maps ρ to ξ the solution to

$$\int_{-1}^{\xi(t)} c(\rho(t,x))\,\mathrm{d}x = \int_{\xi(t)}^{1} c(\rho(t,x))\,\mathrm{d}x$$

Definition (Solution to generalized Hughes' model)

Consider $\mathcal{I} : L^1((0,T)\times\mathbb{R}) \longrightarrow \mathcal{C}^0$. We say that (ρ,ξ) is a solution to the generalized Hughes' model if

- ρ is a solution in the sense of the definition without interface condition
- $\xi = \mathcal{I}(\rho)$ in $\mathcal{C}^0([0,T])$

Saying (ρ,ξ) is a solution to the generalised Hughes' model is equivalent to say that $\rho = \mathcal{S}_0 \circ \mathcal{I}(\rho)$ i.e. ρ is a fixed point of $\mathcal{S}_0 \circ \mathcal{I}$.

Assumptions :

- Let $\rho_0 \in L^1(\mathbb{R})$.
- Let B a convex closed bounded subset of $L^1((0,T)\times\mathbb{R})$
- The operator

$$\mathcal{I} : (B, ||\cdot||_{L^1((0,T)\times\mathbb{R})}) \longrightarrow \quad (\mathcal{C}^0([0,T],\mathbb{R}), ||\cdot||_\infty)$$

  is continuous.
- Assume that f verifies (non-degen):

$$\forall b \in \mathbb{R}, \text{ meas}\{x \in [-||\rho||_\infty; |\rho||_\infty] \text{ s.t. } f'(x) = b\} = 0 \qquad \text{(non-degen)}$$

Corollary (Main result)

If there exists $r > 0$ such that :

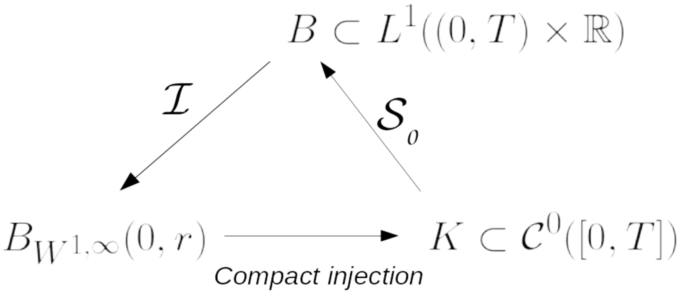
- $\mathcal{I}(B) \subset B_{W^{1,\infty}}(0, r)$
- $\forall \xi \in B_{W^{1,\infty}}(0, r)$, *the unique admissible solution to*

$$\rho_t + [\text{sign}(x - \xi(t))f(\rho)]_x = 0$$

is in B.

then there exists (ρ, ξ) a solution to the generalised Hughes' model.

The proof scheme:



Theorem

Let $\rho_0 \in L^1(\mathbb{R})$*. If* f *satisfies* (non-degen)*, then the solver operator*

$$\mathcal{S}_0 : (W^{1,\infty}((0,T), ||\cdot||_\infty) \longrightarrow (L^1((0,T)\times\mathbb{R}), ||\cdot||_{L^1((0,T)\times\mathbb{R})})$$

is continuous.

This proof relies on the Averaging Compactness Lemma from B. Perthame' book : *Kinetic formulation for hyperbolic conservation law.*

In order to get existence to a solution to a generalised Hughes' model, we have to find B a subset of $L^1((0,T)\times\mathbb{R})$ such that :

- B is a convex closed bounded subset of $L^1((0,T)\times\mathbb{R})$
- The operator

$$\mathcal{I} : (B, ||\cdot||_{L^1((0,T)\times\mathbb{R})}) \longrightarrow (\mathcal{C}^0([0,T],\mathbb{R}), ||\cdot||_\infty)$$

  is continuous.
- $\mathcal{I}(B) \subset B_{W^{1,\infty}}(0,r)$
- $\forall \xi \in B_{W^{1,\infty}}(0,r)$, the unique admissible solution to

$$\rho_t + [\text{sign}(x-\xi(t))f(\rho)]_x = 0$$

  is in B.

For the turning curve problem :

$$\begin{cases} \rho_t + [\operatorname{sign}(x - \xi(t))\rho v(\rho)]_x = 0 \\ \displaystyle\int_{-1}^{\xi(t)} c(\rho(t,x))\,\mathrm{d}x = \int_{\xi(t)}^{1} c(\rho(t,x))\,\mathrm{d}x. \end{cases}$$

with $c(\rho) = 1 + \alpha\rho$, $\alpha > 0$.

There exists $C > 0$ such that any $\rho \in \mathcal{S}_0(W^{1,\infty}((0,T)))$ verifies:

$$\forall a, b \in \mathbb{R}, \forall s, t \in [0,T], \left| \int_a^b \rho(t,x) - \rho(s,x) \, \mathrm{d}x \right| \leq C|t-s|. \quad (8)$$

We call this property the **weak continuity** of ρ.

Then we construct:

$$B_1 = \left\{ \rho \in B_{L^1}(0, T\,||\rho_0||_{L^1}) \text{ s.t. } 0 \leq \rho \leq 1 \text{ and } \rho \text{ verifies } (8) \right\}.$$

This set satisfies all the required properties in order to apply our main result.

Ideas of the proof :

Set $\underline{\xi} = \min(\xi(t), \xi(s))$ and $\bar{\xi} = \max(\xi(t), \xi(s))$.

$$
\begin{aligned}
& 2\,|\xi(t) - \xi(s)| \\
& \leq \left| \int_{-1}^{\underline{\xi}} c(\rho(t,x)) - c(\rho(s,x))\,\mathrm{d}x - \int_{\bar{\xi}}^{1} c(\rho(t,x)) - c(\rho(s,x))\,\mathrm{d}x \right| \\
& \leq \alpha \left| \int_{-1}^{\underline{\xi}} \rho(t,x) - \rho(s,x)\,\mathrm{d}x \right| + \alpha \left| \int_{\bar{\xi}}^{1} \rho(t,x) - \rho(s,x)\,\mathrm{d}x \right| \\
& \leq 2\alpha C |t-s|
\end{aligned}
$$

However, we didn't succeed in proving that $\mathcal{I}(B_1) \subset W^{1,\infty}$ in the case of a general $\mathcal{C}^1$ cost satisfying:

$$\begin{cases} c \in \mathcal{C}^0([0,1]), \\ \forall \rho \in [0,1], c(\rho) \geq 1, \\ c \text{ is increasing on } [0,1]. \end{cases}$$

$$\begin{aligned} &2\left|\xi(t) - \xi(s)\right| \\ &\leq \left(\frac{\bar{\alpha} + \underline{\alpha}}{2}\right) \left| \int_{-1}^{\xi} \rho(t,x) - \rho(s,x)\,\mathrm{d}x - \int_{\bar{\xi}}^{1} \rho(t,x) - \rho(s,x)\,\mathrm{d}x \right| \\ &+ \left(\frac{\bar{\alpha} - \underline{\alpha}}{2}\right) \int_{-1}^{1} \left|\rho(t,x) - \rho(s,x)\right| \,\mathrm{d}x \end{aligned}$$

But, if we set:

$$\mathcal{R}[\rho(\cdot,x)](t) := \delta \int_{-\infty}^{t} \rho(s,x) e^{-\delta(t-s)}\, \mathrm{d}s, \tag{9}$$

we can use B_1 to prove the existence of a solution to:

$$\begin{cases} \rho_t + [\mathrm{sign}(x-\xi(t))\rho v(\rho)]_x = 0 \\ \int_{-1}^{\xi(t)} c(\mathcal{R}[\rho(\cdot,x)](t))dx = \int_{\xi(t)}^{1} c(\mathcal{R}[\rho(\cdot,x)](t))dx, \end{cases}$$

in the $\mathcal{C}^1$ cost case.

With our main result, we can also solve the original turning curve problem with relaxed equilibrium:

$$\begin{cases} \epsilon\dot{\xi}(t) = \int_{\xi(t)}^{1} c(\rho(t,x))dx - \int_{-1}^{\xi(t)} c(\rho(t,x))dx \\ \int_{\xi(0)}^{1} c(\rho(t,x))dx - \int_{-1}^{\xi(0)} c(\rho(t,x))dx = 0. \end{cases}$$

still with $\mathcal{C}^1$ cost.

In this case,

$$B_2 = \{\rho \in B_{L^1}(0, T\,||\rho_0||_{L^1}) \text{ s.t. } 0 \ \leq \rho \leq 1\}$$

satisfies all the required conditions.

Fix $\xi \in W^{1,\infty}$ again.
We want to consider the following dynamic for ρ:

$$\begin{cases} \rho_t + [\mathrm{sign}(x - \xi(t)) f(\rho)]_x = 0 \\ f(\rho(t,1)) \leq g_1 \left(\int_\alpha^1 w_1(x) \rho(t,x) \, \mathrm{d}x \right) \\ \rho(0,\cdot) = \rho_0(\cdot). \end{cases}$$

We call it the constrained exit problem.

Definition (Notion of solution in the constrained case)

$$Q_1(t) := g_1\left(\int_{[\alpha,1]} w_1(x)\rho(t,x)\,\mathrm{d}x\right)$$

A weak solution $\rho \in L^1((0,T)\times\mathbb{R})$ is a solution if:

- For all positive $\phi \in \mathcal{C}^\infty(\{x > \xi(t)\})$, for all $k \in \mathbb{R}$,

$$-\iint_{(0,T)\times\mathbb{R}} |\rho - k|\,\phi_t + q(\rho,k)\phi_x\,\mathrm{d}t\,\mathrm{d}x$$
$$-2\int_0^T \left[1 - \frac{Q_1(t)}{f(\bar{\rho})}\right] f(k)\phi(t,1)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho_0 - k|\phi(0,x)\,\mathrm{d}x \leq 0.$$

- ρ satisfies the classic entropy inequality on $\{x < \xi(t)\}$
- For a.e. $t \geq 0$, $f(\gamma^1_{L,R}\rho(t)) \leq Q_1(t)$.

Assumptions :

$$w_1 \in L^\infty([\alpha, 1], \mathbb{R}^+) \text{ s.t. } \int_\alpha^1 w_1 = 1 \tag{W}$$

$$g_1 \in W^{1,\infty}(\mathbb{R}^+,]0, f(\bar{\rho})]) \text{ is non-increasing.} \tag{G0}$$

From the works:

- R. M. Colombo and P. Goatin, *A well posed conservation law with a variable unilateral constraint*, J. Differ. Equ., 234 (2007), pp. 654–675.
- B. Andreianov, C. Donadello, U. Razafison, and M. D. Rosini, *Riemann problems with non–local point constraints and capacity drop*, (2014).

Corollary

Let $\rho_0 \in L^1(\mathbb{R})$. Let $\xi \in W^{1,\infty}((0,T),]-1,1[)$. There exists a unique solution ρ to the constrained exits problem. The solver operator

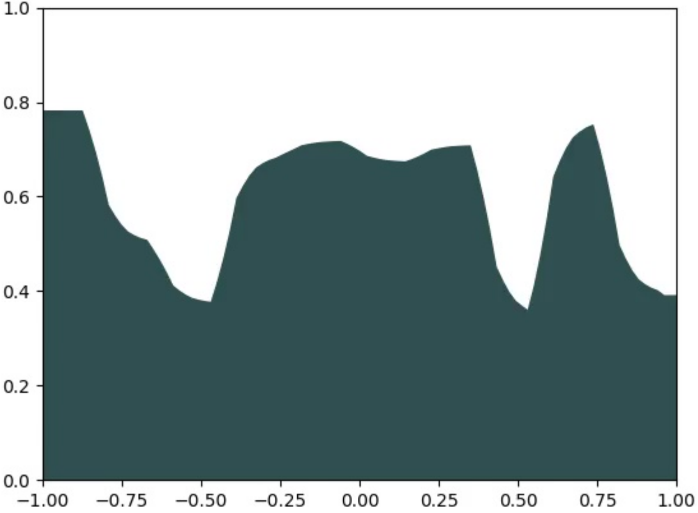
$$\mathcal{S}_g : (W^{1,\infty}((0,T),]-1,1[), ||\cdot||_\infty) \longrightarrow (L^1((0,T)\times\mathbb{R}), ||\cdot||_{L^1})$$

that maps any ξ to the unique solution ρ is continuous.

The proof for uniqueness and continuity follows the same ideas as in the non-constrained case.

Existence of ρ when ξ is fixed : Convergence of a finite volume scheme From the works :

- B. P. Andreianov, C. Donadello, U. Razafison, and M. D. Rosini, *Qualitative behaviour and numerical approximation of solutions to conservation laws with non-local point constraints on the flux and modeling of crowd dynamics at the bottlenecks* (2015).
- B. Andreianov and A. Sylla, *Finite volume approximation and well-posedness of conservation laws with moving interfaces under abstract coupling conditions*. Submitted, (2022)
- A. Sylla, *A lwr model with constraints at moving interfaces*, ESAIM: Mathematical Modelling and Numerical Analysis, 56 (2022)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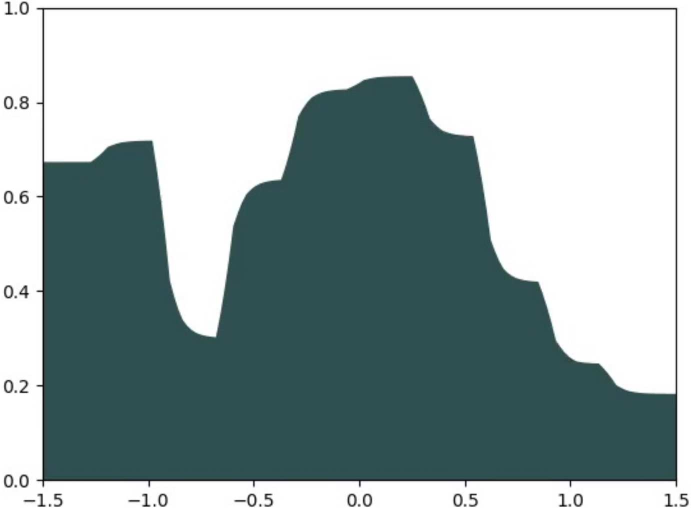
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1.5

A splitting algorithm :

$$\rho_n = \mathcal{FVS}(\xi_n)$$

$$\zeta_{n+1} \text{ solution to } \int_{-1}^{\zeta_{n+1}} c(\rho_n) = \int_{\zeta_{n+1}}^{1} c(\rho_n)$$

$$\xi_{n+1}(s) := \sum_{i=0}^{n} \mathbb{1}_{[i\Delta t,(i+1)\Delta t[}(s) \left(\frac{s - i\Delta t}{\Delta t}\zeta_{i+1} + \frac{(i+1)\Delta t - s}{\Delta t}\zeta_i \right)$$

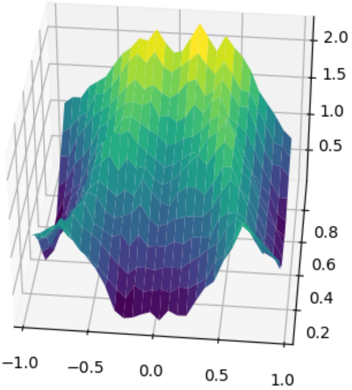
ξ is one step in time ahead of ρ.

In the 2D case, which is the "physical" one :

$$\begin{cases} \rho_t + \operatorname{div}\left[\dfrac{-\nabla\phi}{|\nabla\phi|}\rho v(\rho)\right] = 0 \\ |\nabla\phi| = \dfrac{1}{v(\rho)} \\ \phi(t, x \in E) = 0 \\ \phi(t, x \in \partial\Omega \backslash E) = +\infty. \end{cases}$$

Is there a way to construct a **turning graph** ?

A Fast Marching Method for the Eikonal equation :



Thanks for your attention