Macroscopic pedestrian flows: from Hughes' model to mean field games

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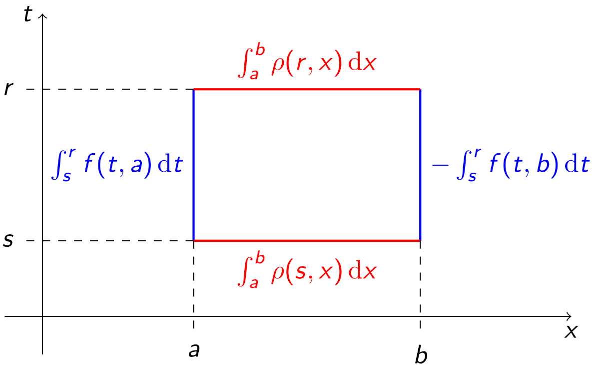
We want to model a moving crowd of pedestrian. The crowd is represented as a pedestrian density $\rho(t, x)$ between 0 and ρ_{max}.

In Zhang and Seyfried 2013, in experimental conditions, $\rho_{max} \simeq 4.5\ m^{-2}$ but the assumed limit is $\rho_{max} = \frac{1}{.113} \simeq 8.84\ m^{-2}$.

Agents in the corridor move towards $+\infty$ with a flux of pedestrian $f(t,x)$.



we have:

$$\int_a^b \rho(r,x)\,\mathrm{d}x = \int_a^b \rho(s,x)\,\mathrm{d}x + \int_s^r f(t,a)\,\mathrm{d}t - \int_s^r f(t,b)\,\mathrm{d}t,$$

$$\int_a^b \int_s^r \partial_t \rho(t,x) + \partial_x f(t,x)\,\mathrm{d}t\,\mathrm{d}x = 0.$$

In LIGHTHILL AND WHITHAM 1955; RICHARDS 1956, the flux is equal to the density multiply by the speed of agents.

$$f(t,x) := \rho(t,x)v(t,x)$$

The velocity v is itself governed by the local density:

$$v(t,x) := v_{\max}\frac{\rho_{max} - \rho}{\rho_{max}}$$

We end up with a **scalar conservation law**,

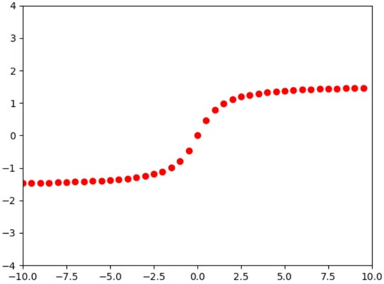
$$\begin{cases} \partial_t \rho(t,x) + \partial_x f(\rho(t,x)) = 0 \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{LWR}$$

We often suppose that $v_{\max} = \rho_{max} = 1$ and recover

$$f(t,x) := f(\rho(t,x)) := \rho(t,x)(1 - \rho(t,x))$$

Non-existence of classical solutions

Method of characteristics for a simple SCL where $f(\rho) = \rho^2/2$ to propagate the initial datum:



Then we consider weak solutions :

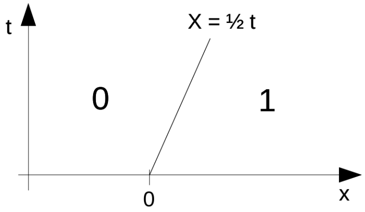
$$\forall \phi \in \mathcal{C}_c^\infty, \quad \iint_{(0,T)\times\mathbb{R}} \rho\phi_t + f(\rho)\phi_x \,\mathrm{d}t\,\mathrm{d}x = 0$$

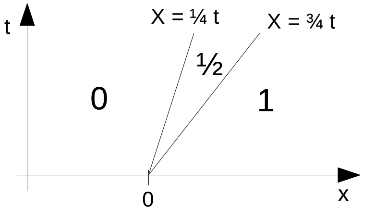
Non-uniqueness of weak solutions

Consider

$$\begin{cases} \rho_t + \left[\rho^2/2\right]_x = 0 \\ \rho(0,x) = \mathbb{1}_{(0,+\infty)} \end{cases}$$

Then the two density functions ρ described below are weak solutions:





Notion of entropy solutions

In Kružkov 1970, the author introduces the notion of entropy solutions.

Definition (Entropy solution to (LWR))

We say that $\rho \in L^\infty((0,+\infty)\times\mathbb{R})$ is an entropy solution to (LWR) if:

- for any non-negative $\phi \in \mathcal{C}_c^\infty((0,+\infty)\times\mathbb{R})$, for any $k \in \mathbb{R}$, we have

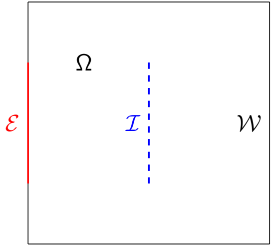
$$\iint_{(0,+\infty)\times\mathbb{R}} |\rho - k|\partial_t\phi + \mathrm{sign}(\rho - k)\,[f(\rho) - f(k)]\,\partial_x\phi \,\mathrm{d}t\,\mathrm{d}x \geq 0,$$

- if we denote by $\rho(0^+, x)$ the strong trace of ρ,

$$\int_{\mathbb{R}} |\rho(0^+, x) - \rho_0(x)|\,\mathrm{d}x = 0.$$

In two dimensions

We consider the following domain:



We assume that the flux of pedestrian is directed by a vector field $\overrightarrow{V}$. Then we get:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\overrightarrow{V}(t,x) f(\rho)) = 0 & x \in \Omega \\ \rho(t,x) = 0 & x \in \mathcal{E} (\text{open-end condition}) \\ \nabla \rho(t,x) \cdot \overrightarrow{n}(x) = 0 & x \in \mathcal{W} (\text{non-crossing condition}). \end{cases} \quad (1)$$

Optimal control problem for pedestrians

Fix $T > 0$. The set of trajectories

$$\Gamma_x := \{\gamma \in W^{1,1}([0,T],\bar{\Omega}) \text{ s.t. } \forall t \in [0,T], \gamma(t) \in \bar{\Omega}, \gamma(0) = x\}$$

We define

- the (optical) distance

$$L(x,y) := \inf_{\substack{\gamma \in \Gamma_x \\ \gamma(T) = y}} \int_0^T c(\gamma(t))|\dot{\gamma}(t)| \, \mathrm{d}t, \qquad (**)$$

  where $c > 0$ is a given running cost function.
- The value function

$$u(x) := \inf_{y \in \mathcal{E}} L(x,y).$$

Hamilton-Jacobi equations

It is classical that an optimal control problem can be seen as an Hamilton-Jacobi equation.

Theorem

Let u be the value function defined before. Then, u is a solution to

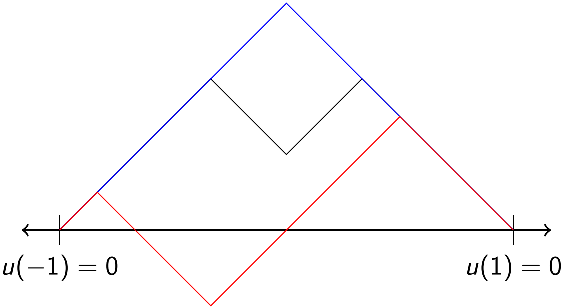
$$\begin{cases} ||\nabla u(x)|| = c(x) & x \in \Omega \\ ||\nabla u(x)|| \leq c(x) & x \in \mathcal{W} \\ u(x) = 0 & x \in \mathcal{E} \end{cases} \tag{Eiko}$$

Non existence of classical solutions

We chose $\Omega = (-1, 1)$ in (Eiko),

$$\begin{cases} |\partial_x u(x)| = 1 & x \in (-1,1) \\ u(x) = 0 & x = \pm 1, \end{cases} \tag{2}$$

Non-existence of $u \in \mathcal{C}^1([-1,1])$ satisfying (2).
Non-uniqueness of solutions almost everywhere:



Viscosity solutions

In Crandall, Evans, and P.-L. Lions 1984 the authors introduce the notion of viscosity solutions for

$$H(x, \nabla u) = 0 \tag{HJ}$$

Definition (Viscosity solutions of (HJ))

We say that $u \in \mathcal{C}(\bar{\Omega})$ is:

- a viscosity subsolution to (HJ) if, for any $\phi \in \mathcal{C}^1(\bar{\Omega})$, for any $x_0 \in \Omega$, if $x \mapsto u(x) - \phi(x)$ admits a maximum at $x = x_0$, we have:

$$H(x, \nabla\phi(x)) \leq 0.$$

- a viscosity supersolution to (HJ) if, for any $\phi \in \mathcal{C}^1(\bar{\Omega})$, for any $x_0 \in \Omega$, if $x \mapsto u(x) - \phi(x)$ admits a minimum at $x = x_0$, we have:

$$H(x, \nabla\phi(x)) \geq 0.$$

- a viscosity solution to (HJ) if it is both a subsolution and a supersolution.

The 2D Hughes model

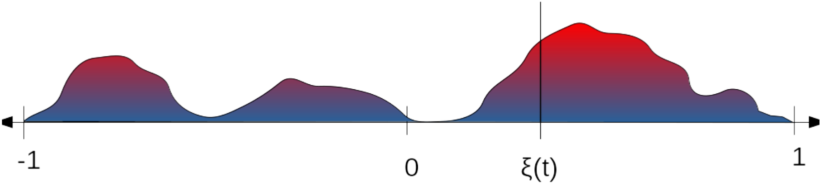
The Hughes' model in two dimensions

$$\begin{cases} \partial_t \rho + \operatorname{div}(\overrightarrow{V}(t,x) f(\rho(t,x))) = 0 & \\ \rho(0,x) = \rho_0(x) & \\ \rho(t,x) = 0 & x \in \mathcal{E} \\ \overrightarrow{V}(t,x) = -\frac{\nabla \phi}{|\nabla \phi|} & \\ |\nabla \phi(t,x)| = c(\rho(t,x)) & x \in \Omega \\ V(t,x) \cdot \overrightarrow{n}(x) = 0 & x \in \mathcal{W} \\ \phi(t,x) = 0 & x \in \mathcal{E}. \end{cases} \tag{Hughes2D}$$

The existence of a solution to (Hughes2D) is still an open question.

In AMADORI AND DI FRANCESCO 2012, the authors reformulate the (Hughes2D) in one dimension as:

$$\begin{cases} \rho_t + [\mathrm{sign}(x-\xi(t))\rho v(\rho)]_x = 0 \\ \rho(0,x) = \rho_0(x) \\ \rho(t, x = \pm 1) = 0 \\ \int_{-1}^{\xi(t)} c(\rho(t,x))\,\mathrm{d}x = \int_{\xi(t)}^{1} c(\rho(t,x))\,\mathrm{d}x \end{cases} \tag{H1D-TC}$$

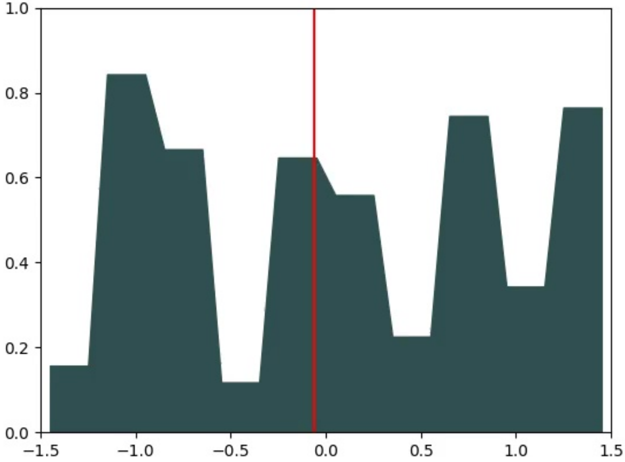


However, proving the equivalence between problem (H1D-TC) *and* (Hughes2D) *is not trivial...*

Existence results

year	authors	strategy of proof	regularity of solutions	main limitation
2014	D. Amadori, P. Goatin, M. D. Rosini	Decoupled	$\rho \in BV$ $\xi \in W^{1,+\infty}$	$\rho(\xi(t)) = 0$
2023	B. Andreianov, M. D. Rosini, G. Stivaletta	Many particles approxiations	$\rho \in L^1 \cap L^\infty$ $\xi \in W^{1,+\infty}$	$c(\rho) = 1 + \alpha\rho$
2024	B. Andreianov, T. Girard	Fixed point approach	$\rho \in L^1 \cap L^\infty$ $\xi \in W^{1,+\infty}$	$c(\rho) = 1 + \alpha\rho$
2024	H. O. Storbrugt	Many particles approxiations	$\rho \in L^1 \cap L^\infty$ $\xi \in BV \cap C^0$	ρ may not be unique
2026	L. Coeuret et. al.	Wave front tracking	$\rho \in BV$ $\xi \in W^{1,\infty}$	$c(\rho) = 1 + \alpha\rho$

Overtaking

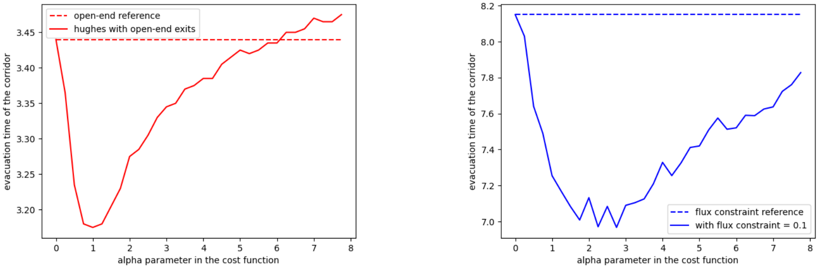


An interesting phenomenon

We compute the evacuation time of the initial datum

$$\rho_0(x) := 0.9 \times \mathbb{1}_{(-0.5,0.95)},$$

for different values of α.

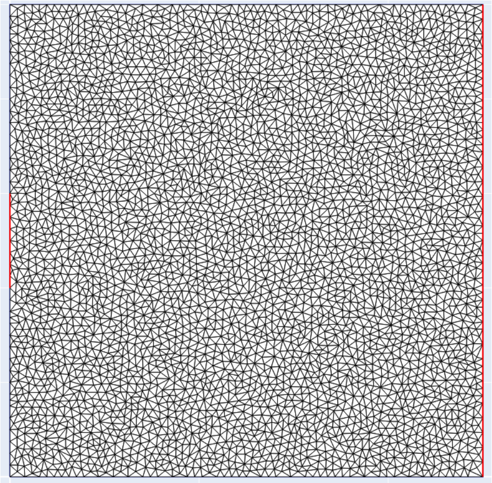


This phenomenon is also observed in Andreianov, Fagioli, Rosini, and Stivaletta 2025 with a different numerical scheme for Hughes' model.

Theoretical issues for existence (Hughes2D)

1. Well-posedness of (SCL) depends on the regularity of V. **Does $V := -\frac{\nabla \phi}{|\nabla \phi|}$ is $\mathcal{C}^0$ away from a finite number of Lipschitz inferfaces ?**
2. Well-posedness of (Eikonal) depends on the regularity of ρ. **Is there a lower semi-continuous function $\rho_*(x)$ such that $\rho(t, x) = \rho_*(x)$ almost everywhere?**
3. Let ρ_1 (resp. ρ_2) be a solution of (SCL) with $V = V_1$ (resp. $V = V_2$). If V_1 and V_2 are close enough, **are ρ_1 and ρ_2 also close in suitable topologies?**
   Lack of compactness...

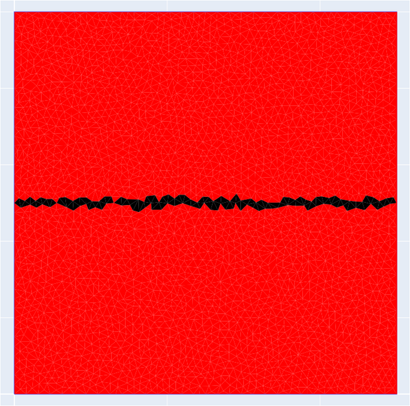
We want to approximate the Monge solution of the Eikonal equation on a triangular mesh $M_\Delta := (\mathcal{T}_n)_{1\leq n\leq N}$.



We discretize the source term:

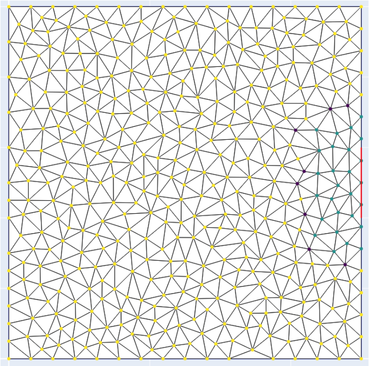
$$\forall n \in [\![1, N]\!], c_n := \inf_{x \in \mathcal{T}_n} c(x), \tag{3}$$

$$c_\Delta(x) := \sum_n \mathbb{1}_{\mathcal{T}_n}(x) c_n.$$

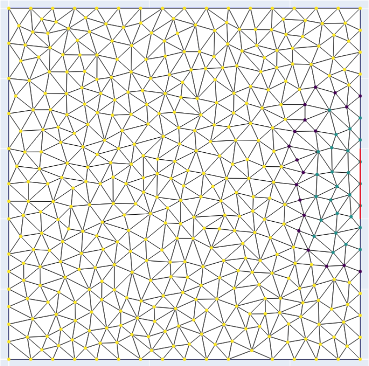


The fast marching principle and the narrow band depth

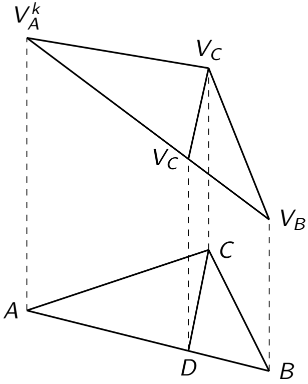
Narrow band depth 1



Narrow band depth 2



We geometrically compute the gradient inside the triangle:



$$V_A^k = \begin{cases} V_B - \frac{\overrightarrow{AB}\cdot\overrightarrow{BC}(V_C - V_B)}{BC^2} + \frac{|\det(\overrightarrow{AB},\overrightarrow{BC})|\sqrt{c_\Delta^2 BC^2 - (V_B - V_C)^2}}{BC^2} & \text{if } c_\Delta |BC| > |V_B - V_C| \\ V_C + c_\Delta\, AC & \text{else} \end{cases}$$

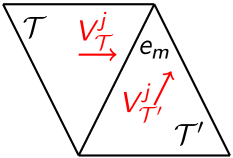
Numerical schemes for viscosity solutions of the eikonal equation

- Fast Marching Finite Differences (FM-FD): Sethian 1996
- FM-FD on triangular mesh: Carlini, Falcone, and Hoch 2013
- FM Semi-Lagrangian (SL): Cristiani and Falcone 2007
- FM-SL for discontinuous source terms: Festa and Falcone 2014

Classical finite volume scheme:

$$\rho_{ABC}^{n+1} = \rho_{ABC}^{n} - \frac{\Delta t}{|ABC|}\left[f_A B|AB| + f_{BC}|BC| + f_{AC}|AC|\right].$$

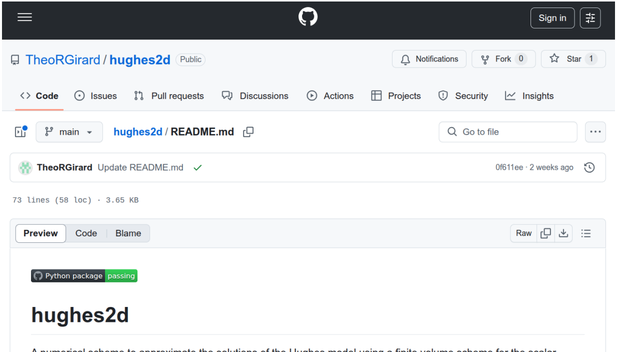
Issue regarding the choice of the flux f:



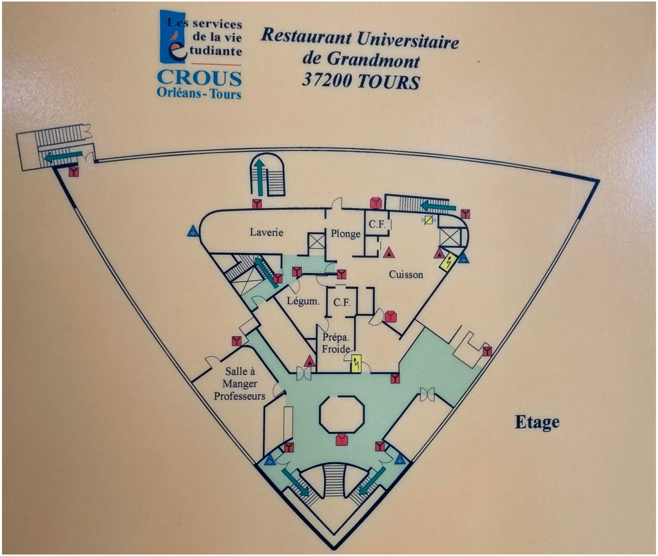
$$\overrightarrow{v}_m(\mathcal{T}) := \frac{V^j\mathcal{T}\rho^j_{\mathcal{T}} + V^j\mathcal{T}'\rho^j_{\mathcal{T}'}}{\left|V^j\mathcal{T}\rho^j_{\mathcal{T}} + V^j\mathcal{T}'\rho^j_{\mathcal{T}'}\right|}. \quad (4)$$

An open source software

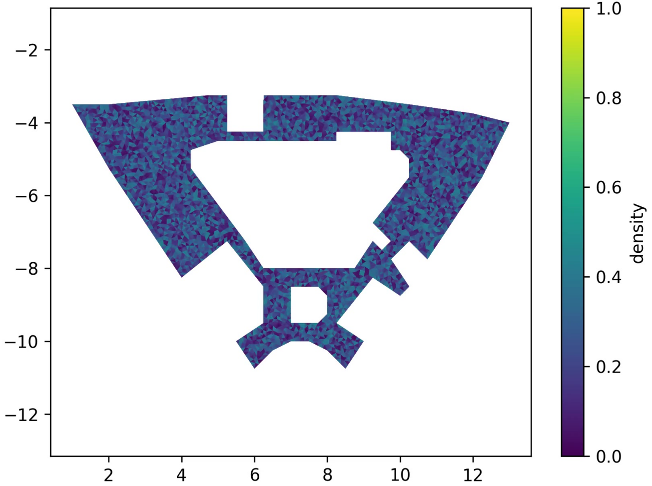
Hughes2D



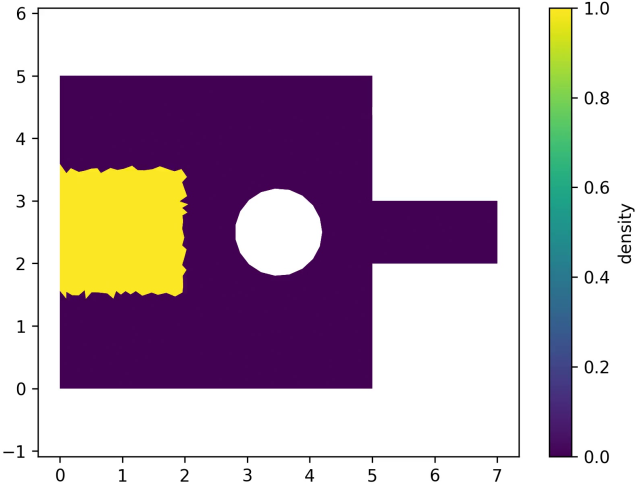
University restaurant



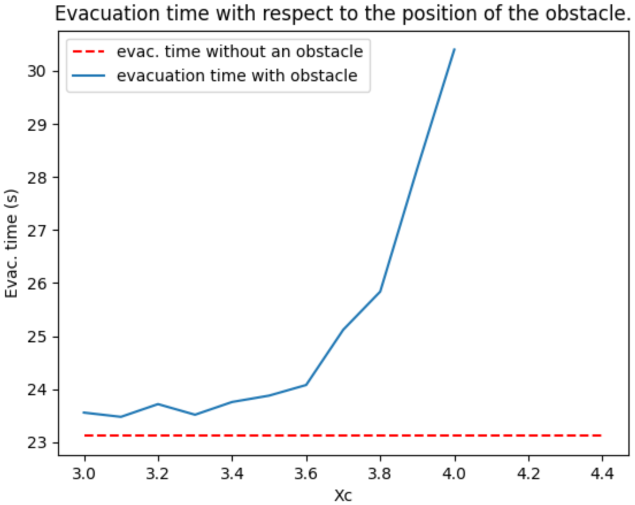
University restaurant



Braess paradox



Braess paradox



Criticism of Hughes' model

- Agents assume that the density will not evolve as time passes (no anticipation).
- Agents react instantaneously to the changes in the density.
- Agents have perfect knowledge of the density at any location at any time → model of Carillo-Martin-Wolfram.
- Agents chose the optimal path at any time and have an infinite computation capacity.

Model of Colombo-Garavello-Lecureux-Mercier

Paper: *Non-local crowd dynamics*, R. M. Colombo, M. Garavello and M. Lécureux-Mercier, C.R.A.S. (2011).

$$\begin{cases} \partial_t \rho + \text{div}\left[\left(\nu(x) + \mathcal{I}[\rho](x)\right) f(\rho)\right] = 0 \\ \nu(x) = -\frac{\nabla \bar{u}(x)}{||\nabla \bar{u}(x)||} \\ ||\nabla \bar{u}(x)|| = 1 \qquad\qquad \text{i.e. } u(x) \simeq d(x, \mathcal{E}) \\ \mathcal{I}[\rho](x) = -\epsilon \frac{\nabla(\rho * \eta)(x)}{\sqrt{1 + ||\nabla(\rho * \eta)(x)||^2}}, \end{cases}$$

where η is a mollifier of radius $r > 0$.

Properties

Good properties:

- wellposed for general assumptions,
- the vector field $\overrightarrow{V}$ is more regular ($\mathcal{I}[\rho](\cdot) \in \mathcal{C}^2(\Omega)$).

Bad properties:

- numerically heavier than Hughes,
- same criticism as Hughes.

Line formation:

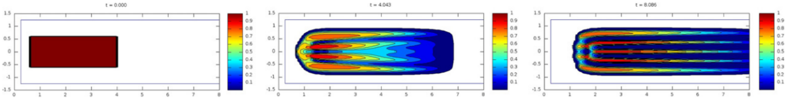
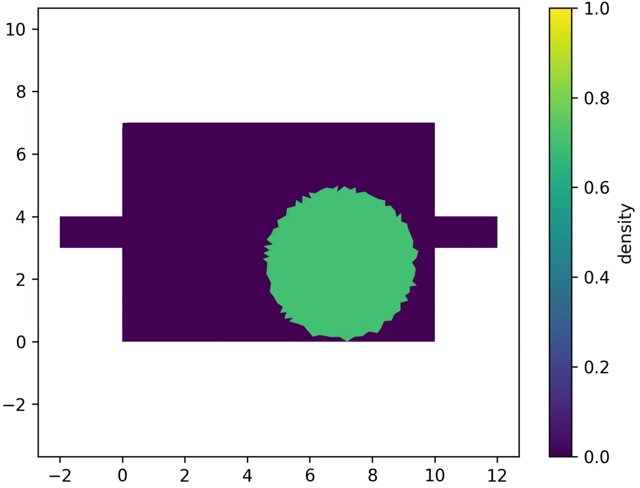


Fig. 1. Solution to (1)–(2)–(4) at time $t = 0, 4.043, 8.086$. Note the formation first of 4 and then of 5 lanes.

Fig. 1. Solution du système (1)–(2)–(4) aux temps $t = 0$, 4,043, 8,086. On remarque la formation de 4 files puis de 5.

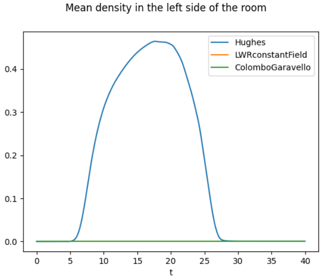
Comparison with Hughes

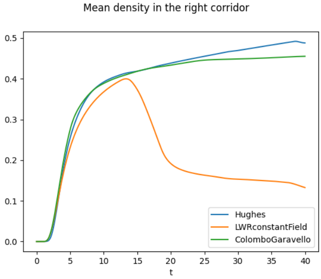
Initial datum:



Comparison with Hughes

Densities in the corridor with respect to time:





Model of Carillo-Martin-Wolfram

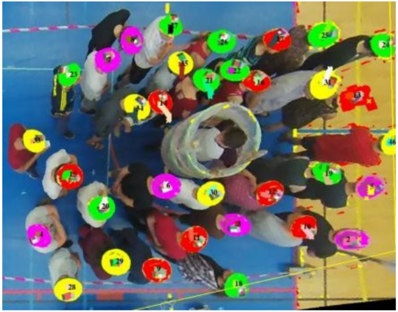
Paper: *An improved version of the Hughes model for pedestrian flow*, J. A. Carillo, S. Martin and M-T. Wolfram, M.M.M.A.S. (2016).

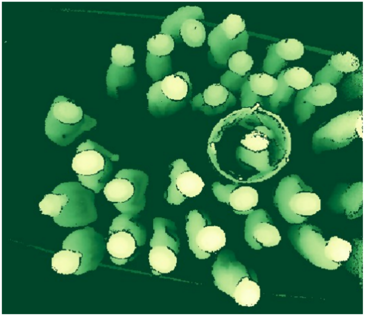
$$\begin{cases} \partial_t \rho + \text{div}\left[\mathcal{P}[\overrightarrow{V}](t,x) f(\rho)\right] = 0 \\ \overrightarrow{V}(x) = -\frac{\nabla_y \bar{u}(x,x)}{||\nabla \bar{u}(x,x)||} \\ ||\nabla_y \bar{u}(x,y)|| = \begin{cases} c(\rho(t,y)) & \text{if } y \in V_x \\ c(\rho_H)) & \text{if } y \in H_x \end{cases}, \end{cases}$$

where $\mathcal{P}$ is a regularizing operator, V_x corresponds to a vision cone of center x, $H_x = \bar{\Omega} \backslash V_x$ and ρ_H corresponds to the density perceived outside the vision cone.

Percefoule

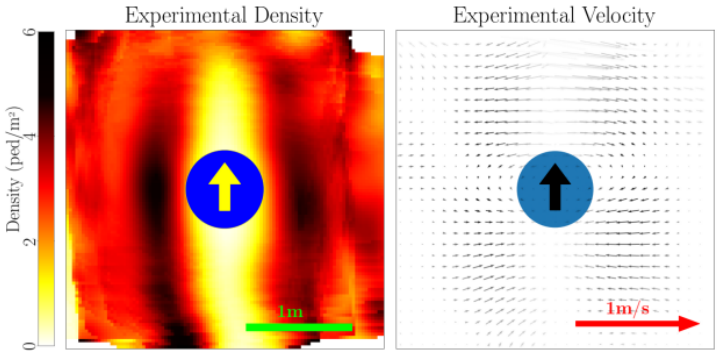
From Nicolas, Kuperman, Ibañez, Bouzat, and Appert-Rolland 2019





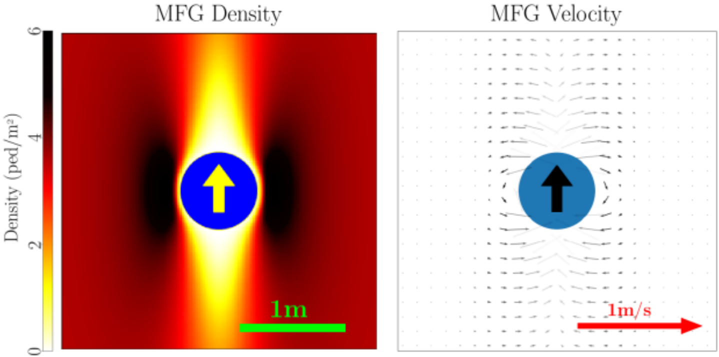
Pedestrian VS grains

From Nicolas, Kuperman, Ibañez, Bouzat, and Appert-Rolland 2019



Anticipation via mean field games

From Butano, Bonnemain, Appert-Rolland, Nicolas, and Ullmo 2024



Mean field games

Most classical MFG system (introduced in Lasry and Lions 2007)

$$\begin{cases} -\partial_t u + H(t,x,\nabla_x u) - \sigma \Delta u = F(x,\rho) \\ u(T,x) = u_T(x) \\ \overrightarrow{V}(t,x) = -\partial_p H(t,x,\nabla u(t,x)) \\ \partial_t \rho + \mathrm{div}(\rho \overrightarrow{V}) - \nu \Delta \rho = 0, \end{cases}$$

where:

- σ corresponds to the noise amount in the choice of the trajecory,
- ν corresponds to the noise amount in the transport of agents,
- $u_T(x)$ is the terminal cost paid by an agent located at x when $T = t$ (as an example $u_T(x) = d(x, \mathcal{E})$ evacuates the domain).

Optimal control problem

If we consider the dynamic

$$\begin{cases} \dot{\gamma}_{x_0}(s) = \alpha(s), \ s \in [t_0, T] \\ \gamma_x(t_0) = x_0, \end{cases}$$

and the minimization problem

$$u(t_0, x_0) = \inf_{\alpha \in A} \int_{t_0}^{T} \frac{1}{2}||\alpha(s)||^2 + c(\rho(s, \gamma_{x_0}(s))) \, \mathrm{d}s + d(\gamma_{x_0}(T), \mathcal{E}),$$

where ρ corresponds to the solution of the Fokker-Planck equation directed by the vector field $-H_p(\nabla u) = -\nabla u$.
Then we obtain the following MFG system:

$$\begin{cases} -\partial_t u + \frac{||\nabla_x u||^2}{2} = c(\rho) \\ u(T, x) = d(x, \mathcal{E}) \\ \overrightarrow{V}(t, x) = -\nabla u(t, x) \\ \partial_t \rho + \mathrm{div}(\rho \overrightarrow{V}) - \nu \Delta \rho = 0. \end{cases}$$

Main differences with Hughes model

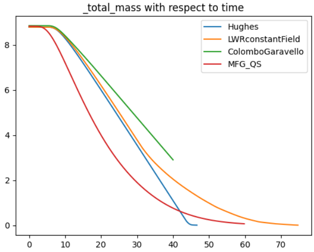
- The strategy of the agents consists in chosing both the direction and the speed of the movement.
- There is no a priori bound on the density ρ.
- A huge class of models with the choices of u_T, H and F.

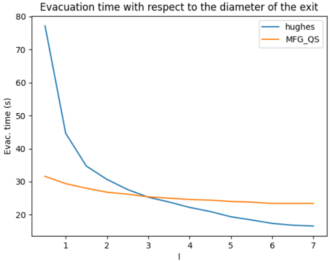
$$\begin{cases} -\partial_t u + \frac{||\nabla_x u||^2}{2} = c(\rho) \\ u(T,x) = d(x,\mathcal{E}) \\ \overrightarrow{V}(t,x) = -\nabla u(t,x) \\ \partial_t \rho + \operatorname{div}(\rho \overrightarrow{V}) - \nu \Delta \rho = 0. \end{cases}$$

Comparison with Hughes

quasi-stationary MFG:

$$\begin{cases} \lambda + H(x, \nabla_x u) - \sigma \Delta u = F(x, \rho) \\ \overrightarrow{V}(t,x) = -\partial_p H(x, \nabla u(t,x)) \\ \partial_t \rho + \operatorname{div}(\rho \overrightarrow{V}) - \nu \Delta = 0, \end{cases}$$



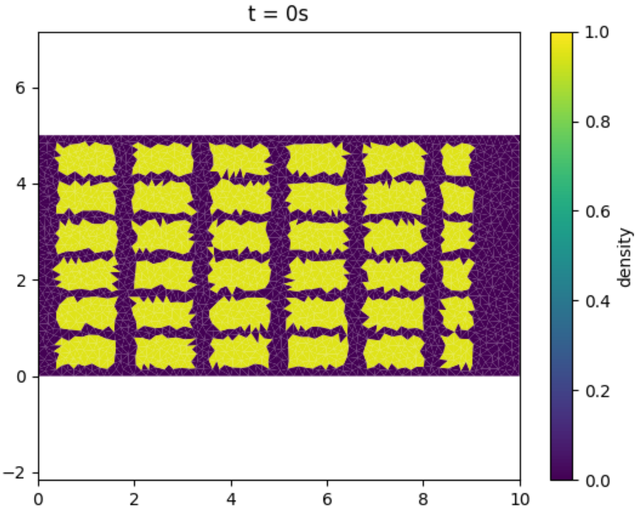


model	Braess	Strategy I	Strategy II	Mushroom	Moving cyl. anticipation
Hughes	No	Slight overshoot	Oscillating	Yes	
CGLM	No	No global strategy	Not oscillating	Yes	
CMW					
MFG_Std		Perfect repartition	Not oscillating		Yes
MFG_QS		Perfect repartition	Small oscillation		

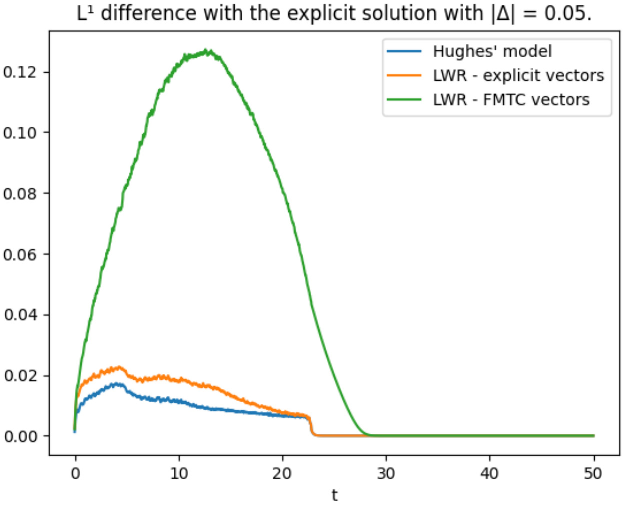
Work in progress with J. Berry and F. Peru...

Merci pour votre attention !

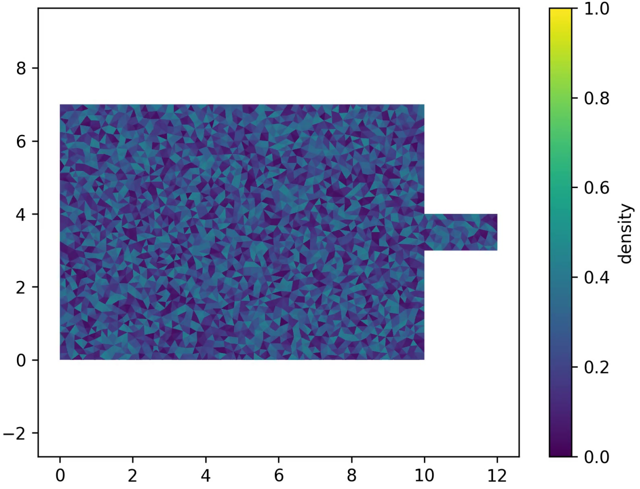
The regularizing effect



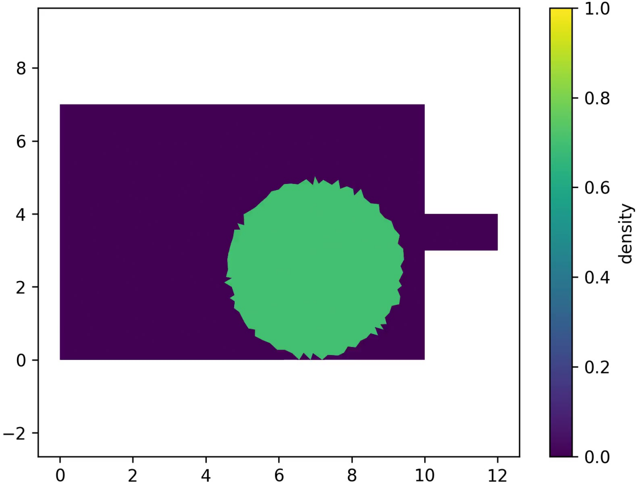
Interesting convergence rate



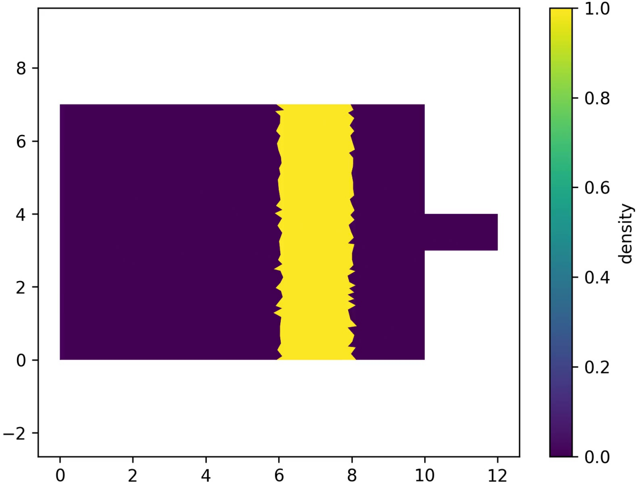
The mushroom property



Different mushrooms



Different mushrooms



Oscillations