Pedestrian crowd models

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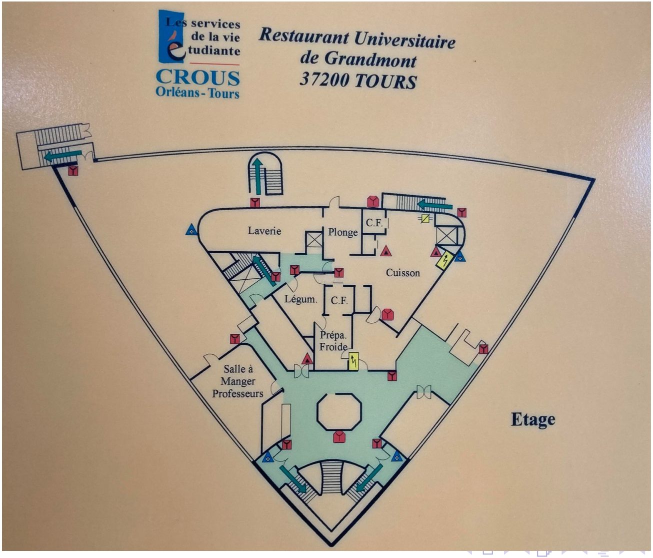
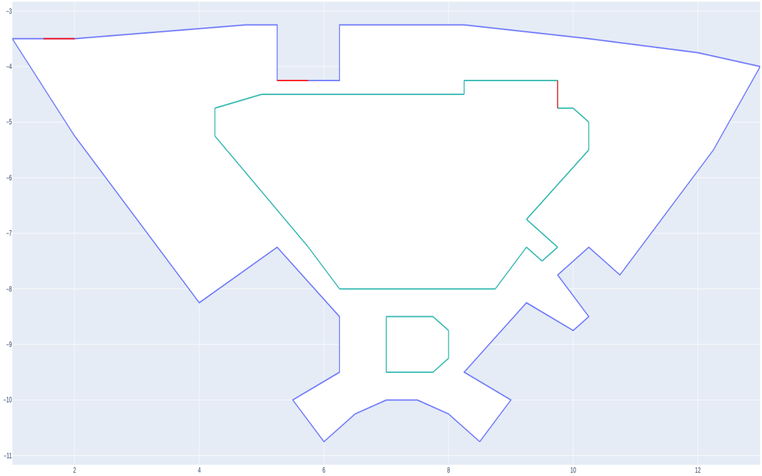
I want to model the university restaurant of Tours in the context of evacuation...



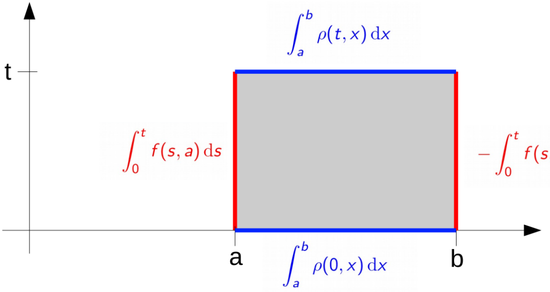
... or its simplified version !



Outline

We want to model a moving crowd. The crowd is represented as a pedestrian density ρ between 0 and 1.

The agents flux is represented by the flux function f.



$$\int_a^b \rho(t,x)\,\mathrm{d}x = \int_a^b \rho(0,x)\,\mathrm{d}x + \int_0^t f(s,a)\,\mathrm{d}s - \int_0^t f(s,b)\,\mathrm{d}s$$

$$\int_a^b \int_0^t \partial_t \rho(s,x)\,\mathrm{d}s\,\mathrm{d}x = -\int_0^t \int_a^b \partial_x f(s,x)\,\mathrm{d}x\,\mathrm{d}s$$

The flux is equal to the density multiply by the speed of agents.

$$f(s,x) := \rho(s,x)v(s,x)$$

The velocity v is itself governed by the local density:

$$v(s,x) := v_{\max}(1-\rho)$$

We set $v_{\max} = 1$ and recover:

$$f(s,x) := f(\rho(s,x)) := \rho(s,x)(1-\rho(s,x))$$

• M. J. Lighthill and G. B. Whitham, On kinematic waves. ii. a theory of traffic flow on long crowded roads, (1955).

We end up with:

$$\int_a^b \int_0^t \partial_t \rho(s,x) + \partial_x f(\rho(s,x)) \,\mathrm{d}x \,\mathrm{d}s = 0$$

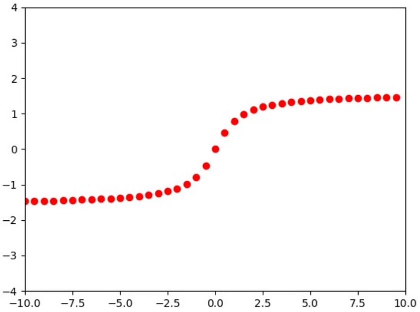
Short version, a scalar conservation law:

$$\rho_t + f(\rho)_x = 0$$

where $f(\rho) = \rho(1-\rho)$. What's known on this equation ?

- **Non-existence of continuous solutions**

We use a method of characteristics to propagate the initial datum:



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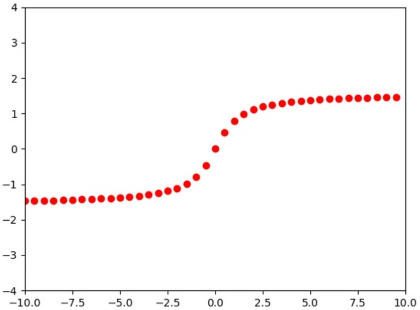


So we consider weak solutions :

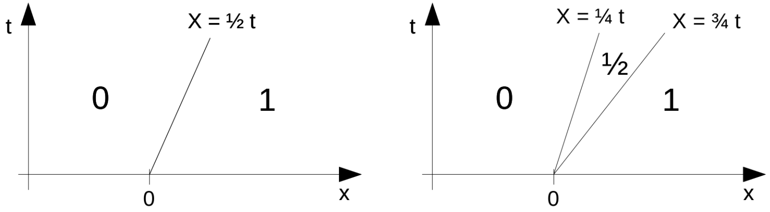
$$\forall \phi \in \mathcal{C}_c^\infty, \quad \iint_{(0,T)\times\mathbb{R}} \rho\phi_t + f(\rho)\phi_x \,\mathrm{d}t\,\mathrm{d}x = 0$$

- **Non-uniqueness of weak solutions**

Consider

$$\begin{cases} \rho_t + \left[\rho^2/2\right]_x = 0 \\ \rho(0,x) = \mathbb{1}_{(0,+\infty)} \end{cases} \qquad (1)$$

Then the two density functions ρ described below are weak solutions:



Krushkov : entropy conditions

We say that $\rho \in L^\infty$ is an entropy solution to

$$\begin{cases} \rho_t + f(\rho)_x = 0 \\ \rho(0,\cdot) = \rho_0(\cdot) \in L^\infty \end{cases}$$

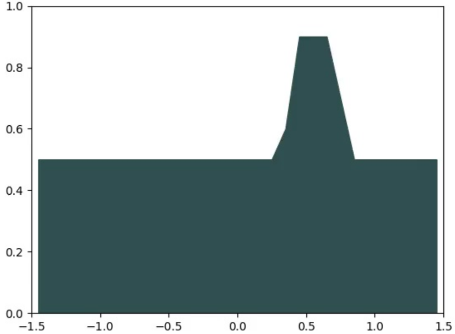
if

$|\rho - k|_t + (\mathrm{sign}(\rho - k)\,(f(\rho) - f(k)))_x \leq 0$ in the distributional sense.

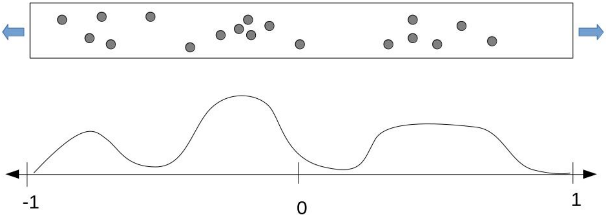
So $\forall k \in \mathbb{R}$, $\forall \phi \in \mathcal{C}_c^\infty$

$$\iint_{(0,T)\times\mathbb{R}} |\rho - k|\phi_t + \mathrm{sign}(\rho - k)\,(f(\rho) - f(k))\,\phi_x \,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} |\rho_0 - k|\phi(0,x)\,\mathrm{d}x \geq 0$$

Interpretation of Kruskov entropy condition in the context of traffic:
The admissible shocks correspond to the traffic jams.



How to model the psychology of crowds ?



A corridor with two doors located at $x = \pm 1$.

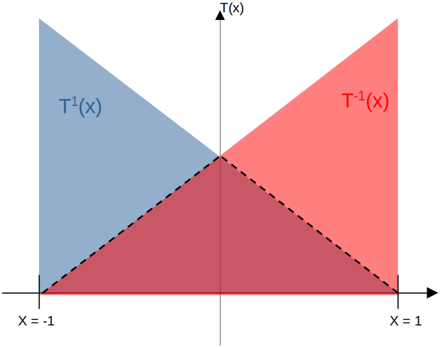
The simplest example : an empty corridor.

We consider a pedestrian located at $x \in [-1, 1]$ at time $t = 0$. This pedestrian computes the time required to reach each of the exits $T^{-1}(x)$ and $T^{1}(x)$. Naturally, the pedestrian will choose the lowest exit time.

If we repeat this process for any x, we can define $u(x)$ the time to exit the corridor if one start at location x:

$$u(x) = \min\{T^{-1}(x), T^{1}(x)\}.$$

If the max speed is 1, we have $T^{-1}(x) = |x + 1|$ and $T^{1}(x) = |x - 1|$.



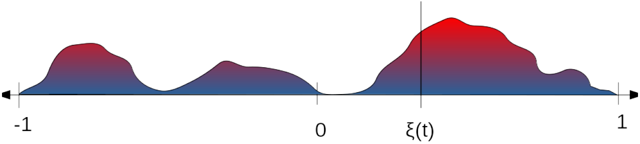
We consider a cost function c depending of the local density. We suppose each agent seeks to minimize not its exit time but its total cost towards the choosen exit. We have again

$$u(x) = \min\{T^{-1}(x), T^{1}(x)\}.$$

But this time, at speed 1 the total cost towards the exit $x = 1$ is

$$T^{1}(x) = \int_{x}^{1} c(\rho(0, y))\,\mathrm{d}y$$

Problem (for later) : the supposed speed of pedestrians is still constant but the speed should vary with the density...

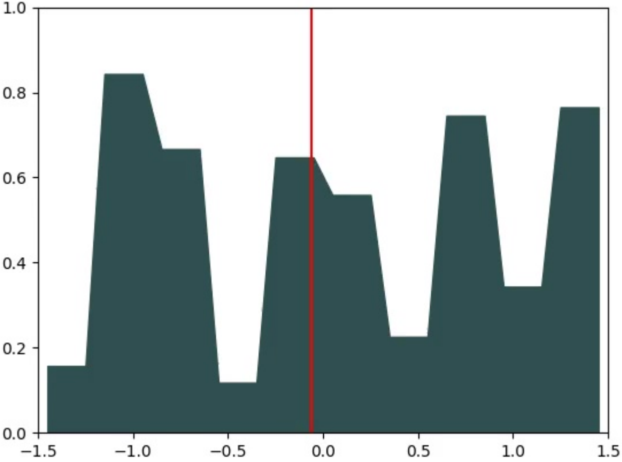


We want to solve:

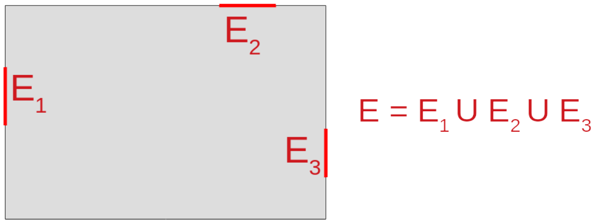
$$\begin{cases} \rho_t + [\text{sign}(x - \xi(t))\rho v(\rho)]_x = 0 \\ \int_{-1}^{\xi(t)} c(\rho(t,x))\,\mathrm{d}x = \int_{\xi(t)}^{1} c(\rho(t,x))\,\mathrm{d}x. \end{cases}$$

The curve ξ is called the turning curve.

In the one-dimensional case, a fixed point argument proves the existence of (ρ, ξ) a solution. (But c has to be affine...)



We want to study the crowd evacuation of an initial density ρ_0 in the room when, at $t = 0$, the agents want to exit the room minimizing their exit time.



Suppose $V(t, x)$ is a vector field corresponding to the choice of direction of an agent located in x at time t. Then the transport equation follows from LWR:

$$\rho_t + \mathrm{div}_x(V(t, x)\rho v(\rho)) = 0$$

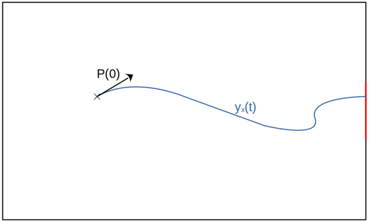
How do we compute V ?

For a fixed and constant density ρ, we use an optimal control problem.

Fix a density ρ in a given domain Ω. Let $\vec{p}(\cdot) \in \mathcal{C}^1([0, +\infty), \mathcal{S}^1)$. We call $\vec{p}$ a "control". Consider the trajectory y_x solution of the Cauchy problem:

$$\begin{cases} \dot{y}_x(t) = v(\rho(y_x(t)))\vec{p}(t) \\ y_x(0) = x. \end{cases}$$

We call all these trajectories the "controlled trajectories" and denote by Y the set of all controlled trajectories.



Now we want to look at the exit time of a pedestrian located at x. If the pedestrian follows the trajectory y_x we compute the total exit time :

$$\int_0^\infty \mathbb{1}_\Omega(y_x(t))\,\mathrm{d}t.$$

Following the 1D case, we can also add a cost function c.

$$J(x, y_x(\cdot)) := \int_0^\infty c(\rho(y_x(t)))\mathbb{1}_\Omega(y_x(t))\,\mathrm{d}t.$$

If we consider that pedestrian always chose the best option to leave the domain, we end up with the following optimisation problem for the total cost:

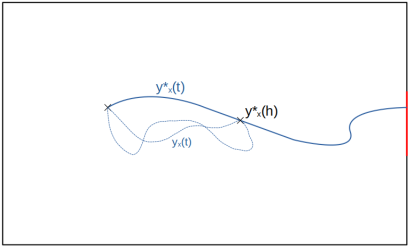
$$u(x) = \inf_{y_x \in Y} J(x, y_x(\cdot)).$$

Suppose that the infinum is a minimum reached for an optimal control $y_x^\star(\cdot)$.

The pedestrian at x should follow the direction field $V(x) = \dot{y}_x^\star(0)$.

Can we compute $\dot{y}_x^\star(0)$ for any x ?
The dynamic programming principle :

$$\forall h > 0, u(x) = \inf_{y_x \in Y} \left\{ \int_0^h c(\rho(y_x(t))) \mathbb{1}_\Omega(y_x(t)) \, \mathrm{d}t + u(y_x(h)) \right\}$$



Notice that, given a $y_x(\cdot)$ and if we suppose that u is differentiable, we have:

$$u(y_x(h)) = u(x) + \int_0^h \nabla u_{y_x(t)} \cdot \dot{y}_x(t)\,\mathrm{d}t.$$

Using both equalities, we get:

$$\inf_{y_x \in Y} \left\{ \int_0^h c(\rho(y_x(t))) + \nabla u_{y_x(t)} \cdot \dot{y}_x(t)\,\mathrm{d}t \right\} = 0.$$

Recall that $c > 0$, if $y_x^\star$ exists, then heuristically we should have

$$\dot{y}_x^\star(0) = -\lambda \nabla u(x).$$

Furthermore,

$$\int_0^h c(\rho(y_x^\star(t))) + \nabla u_{y_x^\star(t)} \cdot \dot{y}_x^\star(t)\,\mathrm{d}t = 0.$$

This should be true for any h and, we get for all t,

$$c(\rho(y_x^\star(t))) + \nabla u_{y_x^\star(t)} \cdot \dot{y}_x^\star(t) = 0.$$

In particular, when $t = 0$, if $\dot{y}_x^\star(0) = -\lambda \nabla u(x)$, we get:

$$||\nabla u(x)|| = \frac{c(\rho(x))}{||\dot{y}_x^\star(0)||} = \frac{c(\rho(x))}{v(\rho(x))}.$$

The Hamilton-Jacobi-Bellman approach:

Solving the optimisation problem $u(x) = \inf_{y_x \in Y} J(x, y_x(\cdot))$
$\Leftrightarrow$ solving the eikonal equation $||\nabla u|| = \frac{c(\rho)}{v(\rho)}$.

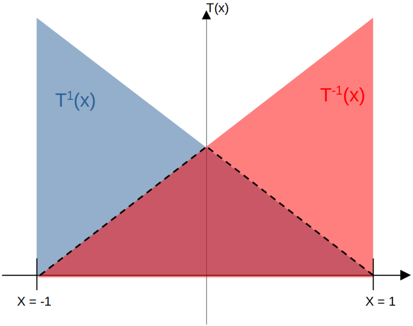
This is more of less the approach of the principle of least action in physics.

For a rigorous proof, Guy and Manu Chasseigne wrote a book where you can find it (and much more) !

Return to one-dimensional case : the solutions of Hamilton-Jacobi equations. If we try to solve the simple case

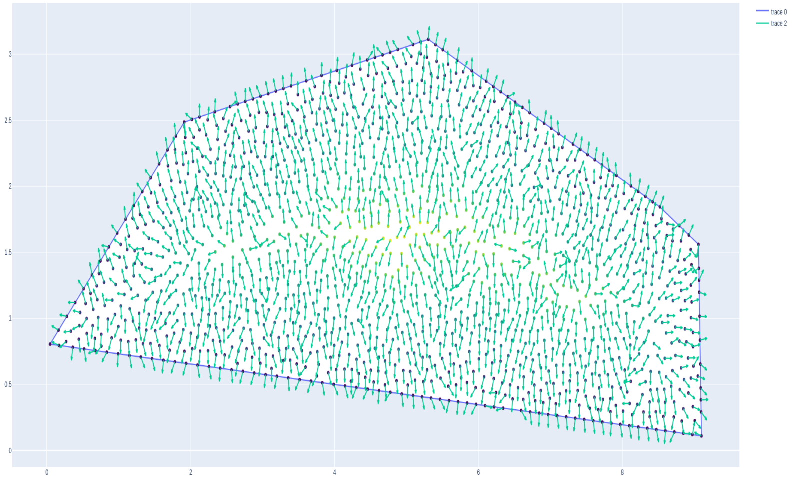
$$\begin{cases} |\partial_x u| = 1 \\ u(x = \pm 1) = 0. \end{cases}$$

Can we say that $|\partial_x u| = 1$ almost everywhere ?

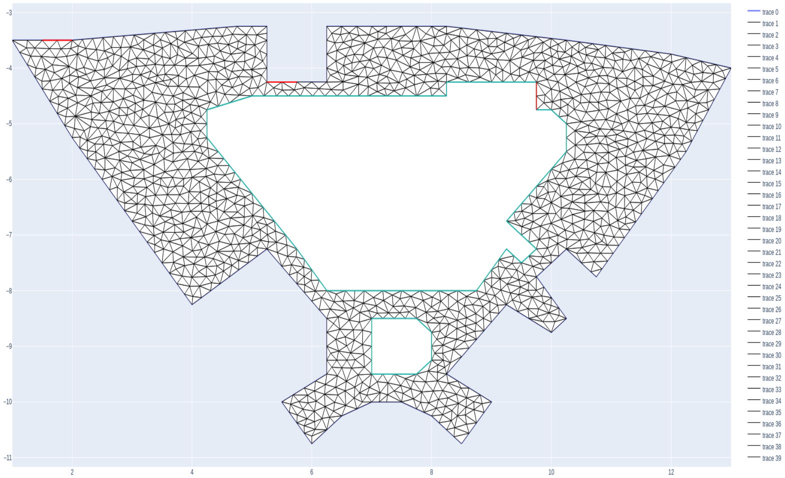


We would lose the uniqueness...
Solution : the notion of viscosity solution.

We can approach the eikonal equation's solution via a fast marching numerical scheme.



The simulation for the university restaurant of Tours:



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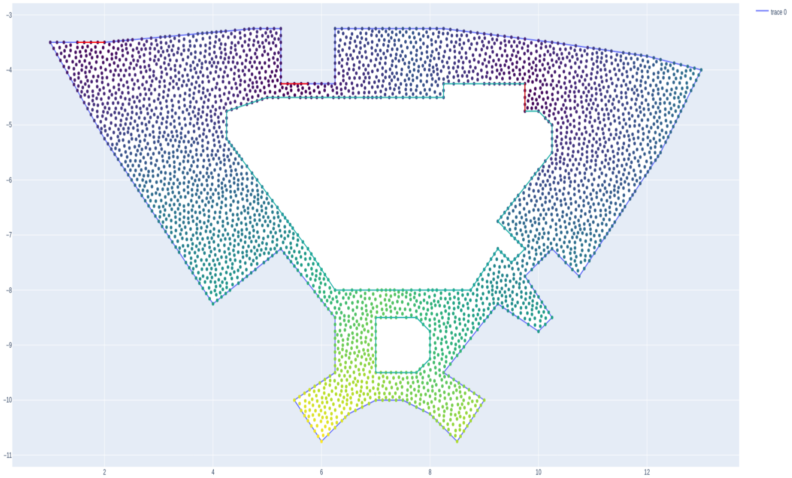
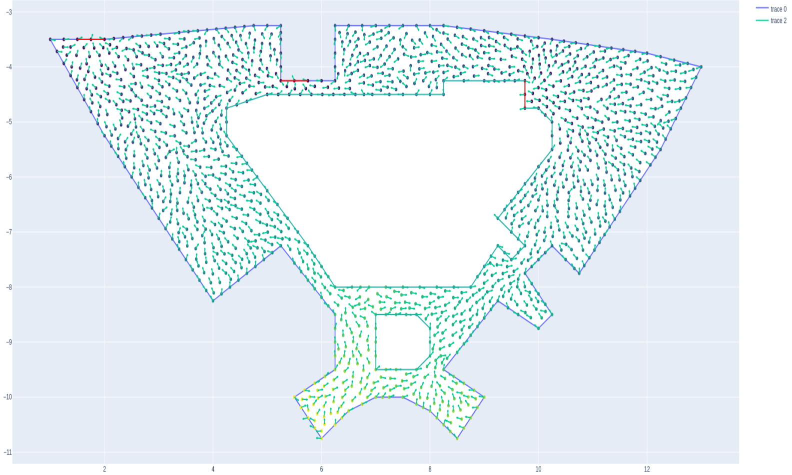
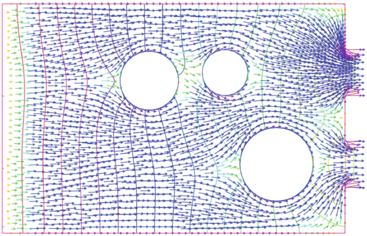


The simulation for the university restaurant of Tours:



To summarize, we should find the solutions of:



2D: directions $\vec{V}$
Gokieli & al.'19

$$\begin{cases} \rho_t + \mathrm{div}_x\left(\frac{-\nabla\phi}{|\nabla\phi|}\rho v(\rho)\right) = 0 \\ |\nabla_x\phi| = \frac{c(\rho)}{v(\rho)} \\ \phi(x\in E) = 0 \\ (\nabla_x\phi\cdot n_D)^+ = 0 \text{ if } x\in\partial D\backslash E \\ \rho(0,x) = \rho(x) \end{cases} \tag{2}$$

where n_D is the normal unit vector to the boundary of the domain D and c is a given cost function depending on the local density.

work in progress...

Thank you.