On the correspondence between Hamilton-Jacobi equations and scalar conservation laws.

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From joint works with P. Cardaliaguet, N. Forcadel and R. Monneau and many discussions with B. Andreianov and V. Perrollaz.

Heuristics

We call scalar conservation laws (SCL) the first order non-linear PDEs of the form:

$$\partial_t \rho + \partial_x H(\rho) = 0.$$

We call Hamilton-Jacobi equations (HJ) the first order non-linear PDEs of the form:

$$\partial_t u + H(\partial_x u) = 0.$$

Notice that, if u is smooth enough, we have:

$$\begin{aligned} \partial_t u + H(\partial_x u) &= 0 \\ \text{then } \partial_x \partial_t u + \partial_x H(\partial_x u) &= 0 \\ \text{then } \partial_t \partial_x u + \partial_x H(\partial_x u) &= 0. \end{aligned}$$

Then, if we denote $\rho := \partial_x u$, we recover the scalar conservation law above.

Meta-definition of an HJ-SCL correspondence

Definition (HJ-SCL correspondence)

We say that we have an **HJ-SCL correspondence** between (HJ) and (SCL) if, for any solution u of

$$\begin{cases} \partial_t u(t,x) + H\left(\partial_x u(t,x)\right) = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{HJ}$$

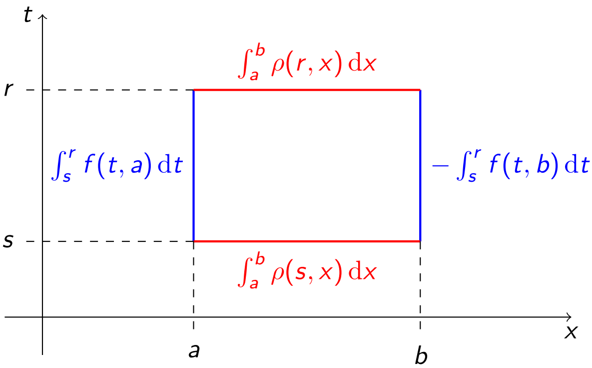
for any solution ρ of

$$\begin{cases} \partial_t \rho(t,x) + \partial_x \left[H(\rho(t,x))\right] = 0 \\ \rho(0,x) = \rho_0(x), \end{cases} \tag{SCL}$$

we have:

$$\rho_0(\cdot) = \partial_x u_0(\cdot) \quad \Rightarrow \quad \forall t,\ \rho(t,\cdot) = \partial_x u(t,\cdot).$$

We consider a quantity $\rho(t,x)$ that is conserved for all times, traveling with a flux $f(t,x) \geq 0$ directed towards the right.



we have:

$$\int_a^b \rho(r,x)\,\mathrm{d}x = \int_a^b \rho(s,x)\,\mathrm{d}x + \int_s^r f(t,a)\,\mathrm{d}t - \int_s^r f(t,b)\,\mathrm{d}t,$$

$$\int_a^b \int_s^r \partial_t \rho(t,x) + \partial_x f(t,x)\,\mathrm{d}t\,\mathrm{d}x = 0.$$

In Lighthill and Whitham 1955; Richards 1956, the flux is equal to the density multiply by the speed of agents.

$$f(t,x) := \rho(t,x)v(t,x)$$

The velocity v is itself governed by the local density:

$$v(t,x) := v_{\max}\frac{\rho_{max} - \rho}{\rho_{max}}$$

We end up with a scalar conservation law,

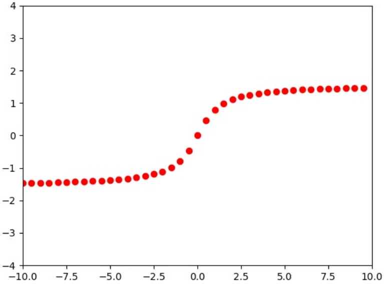
$$\begin{cases} \partial_t\rho(t,x) + \partial_x f(\rho(t,x)) = 0 \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{SCL}$$

We often suppose that $v_{\max} = \rho_{max} = 1$. In order to clarify the notations for the HJ-SCL correspondence we will instead denote

$$H(\rho) := f(\rho).$$

Non-existence of classical solutions

Method of characteristics for a simple SCL where $H(\rho) = \rho^2/2$ to propagate the initial datum:



Then we consider **weak solutions** :

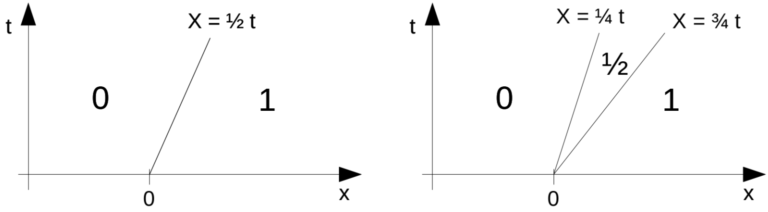
$$\forall \phi \in \mathcal{C}_c^\infty, \quad \iint_{(0,T)\times\mathbb{R}} \rho\phi_t + H(\rho)\phi_x \,\mathrm{d}t\,\mathrm{d}x = 0$$

Non-uniqueness of weak solutions

Consider

$$\begin{cases} \rho_t + \left[\rho^2/2\right]_x = 0 \\ \rho(0,x) = \mathbb{1}_{(0,+\infty)} \end{cases}$$

Then the two functions ρ described below are weak solutions:



Notion of entropy solutions

In KRUŽKOV 1970, the author introduces the notion of entropy solutions.

Definition (Entropy solution to (SCL))

We say that $\rho \in L^\infty((0,+\infty) \times \mathbb{R})$ is an entropy solution to (SCL) if:

• for any non-negative $\phi \in \mathcal{C}_c^\infty((0,+\infty) \times \mathbb{R})$, for any $k \in \mathbb{R}$, we have

$$\iint_{(0,+\infty)\times\mathbb{R}} |\rho - k|\partial_t\phi + \operatorname{sign}(\rho - k)\,[H(\rho) - H(k)]\,\partial_x\phi\,\mathrm{d}t\,\mathrm{d}x \geq 0,$$

• if we denote by $\rho(0^+, x)$ the strong trace of ρ,

$$\int_{\mathbb{R}} |\rho(0^+, x) - \rho_0(x)|\,\mathrm{d}x = 0.$$

Optimal control problem

For any (t, x), we consider controlled trajectories of the form:

$$\gamma_x \in W^{1,\infty}(0,t) \text{ s.t. } \gamma_x(0) = x.$$

We denote by $\Gamma_{(t,x)}$ the set of all admissible trajectories. Then we introduce the cost functional:

$$J(t,x,\gamma_x) := \int_0^t L(\gamma_x(s), \dot{\gamma}_x(s))\, \mathrm{d}s + g(\gamma_x(t)),$$

where:

- $L(x, v)$ corresponds to a running (or instantaneous) cost function,
- g corresponds to a terminal cost.

Hamilton-Jacobi equations

We define the value function

$$u(t,x) := \inf_{\gamma \in \Gamma_{(t,x)}} J(t,x,\gamma). \tag{1}$$

It is classical that an optimal control problem can be seen as an Hamilton-Jacobi equation.

Theorem

Let u be the value function defined by (1). Then, u is a solution to

$$\begin{cases} \partial_t u + H(x, \partial_x u) = 0 \\ u(0,x) = g(x), \end{cases} \tag{HJ}$$

where H is given by:

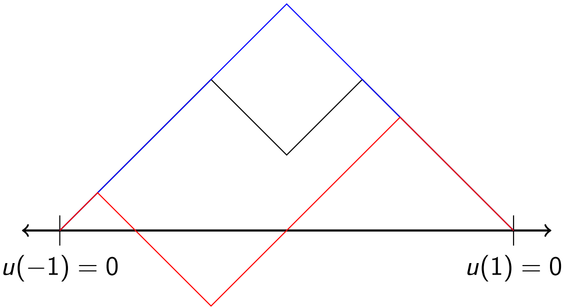
$$H(x,p) := \sup_{v \in \mathbb{R}} \{-pv - L(x,v)\}.$$

Non existence of classical solutions

We chose $\Omega = (-1, 1)$ and the simplest Hamilton-Jacobi equation:

$$\begin{cases} |\partial_x u(x)| = 1 & x \in (-1,1) \\ u(x) = 0 & x = \pm 1, \end{cases} \tag{2}$$

Non-existence of $u \in \mathcal{C}^1([-1,1])$ satisfying (2).
Non-uniqueness of solutions almost everywhere:



Viscosity solutions

In Crandall, Evans, and Lions 1984 the authors introduce the notion of viscosity solutions for (HJ).

Definition (Viscosity solutions of (HJ))

We say that $u \in \mathcal{C}([0,+\infty) \times \mathbb{R})$ is:

- a viscosity subsolution to (HJ) if, for any $\phi \in \mathcal{C}^1([0,+\infty) \times \mathbb{R})$, for any $x_0 \in \Omega$, if $(t,x) \mapsto u(t,x) - \phi(t,x)$ admits a maximum at $(t,x) = (t_0,x_0)$, we have:

$$\partial_t \phi(t_0,x_0) + H(x_0, \partial_x \phi(t_0,x_0)) \leq 0.$$

- a viscosity supersolution to (HJ) if, for any $\phi \in \mathcal{C}^1([0,+\infty) \times \mathbb{R})$, for any $x_0 \in \Omega$, if $(t,x) \mapsto u(t,x) - \phi(t,x)$ admits a minimum at $(t,x) = (t_0,x_0)$, we have:

$$\partial_t \phi(t_0,x_0) + H(x_0, \partial_x \phi(t_0,x_0)) \geq 0.$$

- a viscosity solution to (HJ) if it is both a subsolution and a supersolution.

Theorem (Main results on viscosity solutions)

Let $K > 0$ and let $H(x,p)$ be convex in p for any x and such that:

$$H(x,p) - H(x,q) \leq K(|x|+1)|p-q|.$$

Then we have that:

• *for any initial datum $u_0 \in W^{1,\infty}$, there exists a viscosity solution u of (HJ);*

• *let u (resp. v) be a viscosity subsolution (resp. supersolution), then we have the comparison principle:*

$$u \leq v \ \ on \ (0,+\infty) \times \mathbb{R};$$

• *the value function u defined by (1) is the unique viscosity solution of (HJ) with a terminal cost $g = u_0$.*

Entropy solutions VS viscosity solutions I

Suppose that u is affine by part of the form

$$u(t,x) = -Kt + k_L x \mathbb{1}_{\mathbb{R}^-} + k_R x \mathbb{1}_{\mathbb{R}^+},$$

such that

$$K = H(k_L) = H(k_R).$$

Then, u being a viscosity subsolution means that, for any $p \in [k_L, k_R]$, we have

$$-K + H(p) \leq 0.$$

On the other hand, u being a viscosity supersolution means that, for any $p \in \mathbb{R} \backslash (k_L, k_R)$, we have

$$-K + H(p) \geq 0.$$

Entropy solutions VS viscosity solutions II

Now we consider

$$\rho(t,x) := k_L \mathbb{1}_{\mathbb{R}^-} + k_R \mathbb{1}_{\mathbb{R}^+}.$$

If ρ satisfies the Kruzhkov entropy inequality with $|\cdot - p|$, then we get:

$$\operatorname{sign}(k_L - p)\left[H(k_L) - H(p)\right] \leq \operatorname{sign}(k_R - p)\left[H(k_R) - H(p)\right].$$

In particular, if $p \in [k_L, k_R]$, we get

$$H(p) - K \leq K - H(p) \Leftrightarrow -2K + 2H(p) \leq 0.$$

And if $p \in \mathbb{R}\backslash(k_L, k_R)$, we get

$$K - H(p) \leq -K + H(p) \Leftrightarrow -2K + 2H(p) \geq 0.$$

HJ-SCL correspondence in the litterature

In the litterature:

- For stationary equations: Caselles 1992
- In the homogenous convex case, using vanishing viscosity: Corrias, Falcone, and Natalini 1995
- In the homogenous convex case, using front-tracking: Karlsen and Risebro 2002
- In the heterogenous case, using vanishing viscosity: Colombo, Perrollaz, and Sylla 2023

Theorem (Colombo, Perrollaz, and Sylla 2023)

Let $H(x,p)$ be convex in p for all x, $H \in \mathcal{C}^3 : \mathbb{R} \times \mathbb{R}$, independent of x outside of a compact subset of $\mathbb{R}$. Let $u_0 \in W^{1,\infty}$, we denote $\rho_0 := \partial_x u_0 \in L^\infty$. We denote by u the unique viscosity solution of

$$\begin{cases} \partial_t u + H(x, \partial_x u) = 0 \\ u(0,x) = u_0(x). \end{cases}$$

We denote by ρ the unique entropy solution of

$$\begin{cases} \partial_t \rho + \partial_x H(x, \rho) = 0 \\ \rho(0,x) = \rho_0(x). \end{cases}$$

Then we have, in the distributional sense

$$\rho = \partial_x u.$$

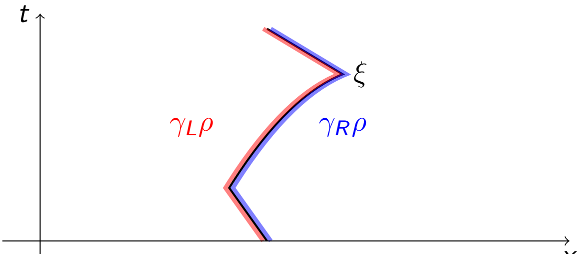
Strong traces of entropy solutions

Theorem (Vasseur 2001; Neves, Evgeniy Panov, and Silva 2018)

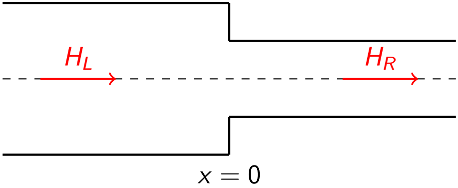
If $\rho \in L^\infty((0,+\infty)\times\mathbb{R})$ is an entropy solution of (SCL), then, for any open subset $\Omega \subset (0,+\infty)\times\mathbb{R}$ with Lipscitz boundary, there exist a **strong trace** *of ρ denoted by*

$$\gamma\rho(\cdot) \in L^\infty(\partial\Omega).$$

If $\xi \in W^{1,\infty}((0,+\infty))$, we denote



Scalar conservation law with discontinuous flux



This situation can be modelled by the **discontinuous-flux scalar conservation law**:

$$\begin{cases} \partial_t \rho(t,x) + \partial_x \left[H_L(\rho(t,x)) \right] = 0 & x < 0 \\ \partial_t \rho(t,x) + \partial_x \left[H_R(\rho(t,x)) \right] = 0 & x > 0 \\ \rho(0,x) = \rho_0(x) & x \in \mathbb{R}. \end{cases} \qquad \text{(SCL-disc)}$$

The notion of $\mathcal{G}$-entropy solutions

Definition ($\mathcal{G}$-entropy solution Andreianov, Karlsen, and Risebro 2011)

Let $\rho_0 \in L^\infty(\mathbb{R})$. Let $\mathcal{G} \subset \mathbb{R}^2$ be a germ. We say that $\rho \in L^\infty((0,T) \times \mathbb{R})$ is a $\mathcal{G}$-entropy solution to (SCL-disc) if:

- ρ is a weak solution to (SCL-disc).
- ρ satifies the Kruzhkov entropy inequalities on $(-\infty, 0)$ with H_L and on $(0, +\infty)$ with H_R.
- The strong traces satisfy the condition

$$(\gamma_L \rho(t), \gamma_R \rho(t)) \in \mathcal{G} \quad \text{for a.e. } t \in (0,T).$$

The notion of L^1 dissipative germ

Definition (Germ and properties)

i) (germ) A set $\mathcal{G} \subset \mathbb{R}^2$ is a germ if any element (p_L, p_R) of $\mathcal{G}$ satisfies the Rankine-Hugoniot condition, i.e. the conservation of the density:

$$H_L(p_L) = H_R(p_R).$$

ii) (L^1-dissipative germ) The germ $\mathcal{G}$ is L^1-dissipative if any pair of elements of $\mathcal{G}$ satisfies the L^1 dissipativity condition i.e. for any $p = (p_L, p_R), q = (q_L, q_R) \in \mathcal{G}$, we have:

$$\text{sign}(p_L - q_L)\left[H_L(p_L) - H_L(q_L)\right] \geq \text{sign}(p_R - q_R)\left[H_R(p_R) - H_R(q_R)\right].$$

iii) (maximal L^1-dissipative germ) The germ $\mathcal{G}$ is maximal in the sense of the inclusion.

iv) (complete germ) The germ $\mathcal{G}$ is complete if there exists a $\mathcal{G}$ entropy solution for any initial datum constant on each half-line.

Well-posedness of (SCL-disc)

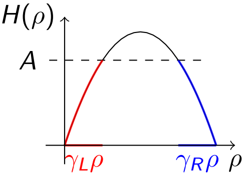
Theorem (Existence and uniqueness for (SCL-disc), Andreianov, Karlsen, and Risebro 2011)

Let $\rho_0 \in L^\infty(\mathbb{R})$. Let $\mathcal{G}$ be a germ.

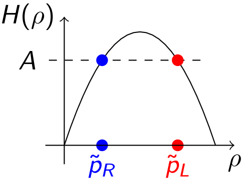
(*i*) *If the germ $\mathcal{G}$ is* **L^1-dissipative** *and* **maximal**, *there exists at most one $\mathcal{G}$-entropy solution to* (SCL-disc).

(*ii*) *If the germ $\mathcal{G}$ is* **complete**, *then there exists a solution $\mathcal{G}$-entropy to* (SCL-disc).

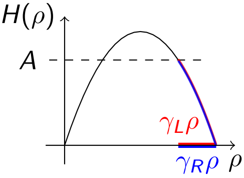
$\mathcal{G}_A := \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$



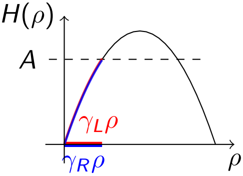
$\mathcal{G}_1 := [0, 1/4] \times [3/4, 1]$



$\mathcal{G}_2 := \{(\tilde{p}_L, \tilde{p}_R)\}$



$\mathcal{G}_3 := [3/4, 1] \times [3/4, 1]$



$\mathcal{G}_4 := [0, 1/4] \times [0, 1/4]$

Discontinuous Hamilton-Jacobi equations

A prototype example of **discontinuous Hamilton-Jacobi equation**:

$$\begin{cases} \partial_t u + H_L(x, \partial_x u) = 0 & x < 0 \\ \partial_t u + H_R(x, \partial_x u) = 0 & x > 0 \\ u(0, x) = u_0(x). \end{cases} \tag{HJ-disc}$$

Variations of the notion of viscosity solutions for discontinuous Hamilton-Jacobi equations:

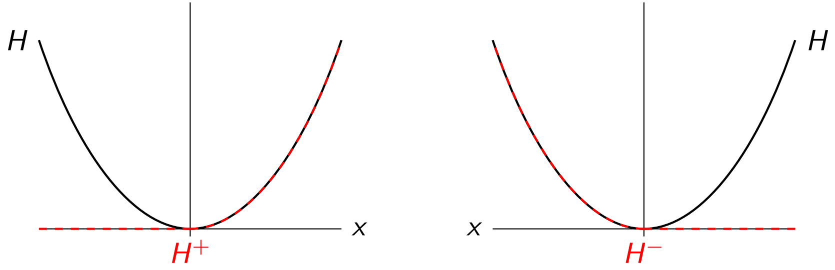
- Weak viscosity solutions, Ishii 1985
- Flux-limited solutions, Imbert and Monneau 2017
- An overview on the subject, Barles and Chasseigne 2023

Notations for the flux-limited approach

Let $H : \mathbb{R} \to \mathbb{R}$ be strictly convex and be such that $H(p) \xrightarrow[|p|\to+\infty]{} +\infty$. We denote $b := \operatorname{argmin} H$. We define the two associated monotone envelopes for any $p \in \mathbb{R}$

$$H^+(p) = \begin{cases} H(b) & \text{for} \quad p \in (-\infty, b] \\ H(p) & \text{for} \quad p \in [b, +\infty) \end{cases}, \quad H^-(p) = \begin{cases} H(p) & \text{for} \quad p \in (-\infty, b] \\ H(b) & \text{for} \quad p \in [b, +\infty) \end{cases}$$

that we represent below for given example of H below.



The flux-limited approach

In Imbert and Monneau 2017, the authors introduce the notion of flux-limited viscosity solution for **discontinuous time-dependent Hamilton-Jacobi equations** (HJ-disc).

Definition (Strong flux-limited solution, Imbert and Monneau 2017)

Let $A \in \mathbb{R}$. Let $H_{L,R}$ be convex. We define, for any $(p_L, p_R) \in \mathbb{R}^2$, the following Hamiltonian

$$H_A(p_L, p_R) := \max\{A, H_L^+(p_L), H_R^-(p_R)\}.$$

Let $u \in \mathcal{C}^0((0,+\infty) \times \mathbb{R})$. We say that u is an **A-flux-limited viscosity solution** to (HJ-disc) if u is a viscosity solution of

$$\begin{cases} u_t(t_0, x_0) + H_L(u_x(t_0, x_0)) = 0 & \text{if } x_0 < 0 \\ u_t(t_0, x_0) + H_R(u_x(t_0, x_0)) = 0 & \text{if } x_0 > 0 \\ u_t(t_0, x_0) + H_A(u_x(t_0, x_0^-), u_x(t_0, x_0^+)) = 0 & \text{if } x_0 = 0. \end{cases}$$

From Imbert and Monneau 2017,

Theorem (Well-posedness of (HJ-disc))

For any $A \in \mathbb{R}$, there exist a unique A-flux-limited viscosity solution of (HJ-disc).

HJ-SCL correspondence with a discontinuity

Theorem (Cardaliaguet, Forcadel, Girard, and Monneau 2024)

Let $u_0 \in W^{1,\infty}(\mathbb{R})$ and $\rho_0 := \partial_x u_0$. Let $H_{L,R}$ be two convex and coercive functions. Under suitable assumptions on u_0 and $H_{L,R}$, for any $A \in \mathbb{R}$, we construct the following germ:

$$\mathcal{G}_A := \left\{(p_L, p_R) \in \mathbb{R}^2 \ s.t. \ H_L(p_L) = H_R(p_R) = H_A(p_L, p_R)\right\}.$$

Then, if we denote by:

- *u the unique* **A-flux-limited solution** *to* (HJ-disc)*;*
- *ρ the unique* **$\mathcal{G}_A$-entropy solution** *to* (SCL-disc)*;*

then, in the distributional sense, we have

$$\partial_x u = \rho.$$

Idea of the proof I

Let $\mathsf{F}_{L,R}(\cdot,\cdot)$ define a numerical flux (such as Godunov flux) adapted to $H_{L,R}$. We define the following finite differences scheme for (HJ-disc):

$$\frac{u_\Delta(t+\Delta t,x)-u_\Delta(t,x)}{\Delta t}$$
$$= -\begin{cases} \mathsf{F}_L\left(\frac{u_\Delta(t,x+\Delta x)-u_\Delta(t,x)}{\Delta x}, \frac{u_\Delta(t,x)-u_\Delta(t,x-\Delta x)}{\Delta x}\right) & \text{if } x<0 \\ \mathsf{F}_R\left(\frac{u_\Delta(t,x+\Delta x)-u_\Delta(t,x)}{\Delta x}, \frac{u_\Delta(t,x)-u_\Delta(t,x-\Delta x)}{\Delta x}\right) & \text{if } x>0 \\ H_A\left(\frac{u_\Delta(t,x+\Delta x)-u_\Delta(t,x)}{\Delta x}, \frac{u_\Delta(t,x)-u_\Delta(t,x-\Delta x)}{\Delta x}\right) & \text{if } x=0. \end{cases}$$

Then, we denote

$$p_\Delta(t,x+\Delta x/2) = \frac{u_\Delta(t,x+\Delta x)-u_\Delta(t,x)}{\Delta x}.$$

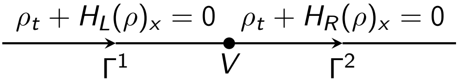
Idea of the proof II

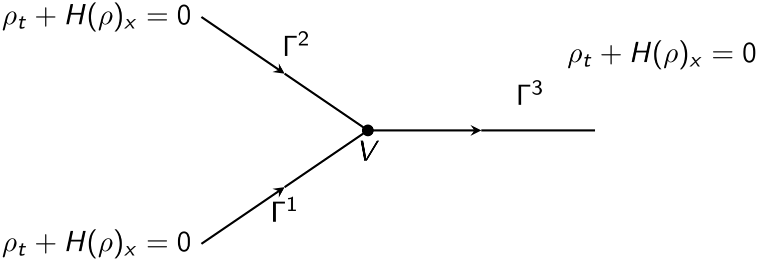
Then we obtain a finite volume scheme for (SCL-disc):

$$\frac{\rho_\Delta(t+\Delta t,x)-\rho_\Delta(t,x)}{\Delta t}$$
$$= -\frac{1}{\Delta x}\begin{cases} \mathsf{F}_L(\rho_\Delta(t,x-\Delta x),\rho_\Delta(t,x)) - \mathsf{F}_L(\rho_\Delta(t,x),\rho_\Delta(t,x+\Delta x)) & \text{if } x<-\Delta x/2 \\ \mathsf{F}_L(\rho_\Delta(t,x-\Delta x),\rho_\Delta(t,x)) - H_A(\rho_\Delta(t,x),\rho_\Delta(t,x+\Delta x)) & \text{if } x=-\Delta x/2 \\ H_A(\rho_\Delta(t,x-\Delta x),\rho_\Delta(t,x)) - \mathsf{F}_R(\rho_\Delta(t,x),\rho_\Delta(t,x+\Delta x)) & \text{if } x=\Delta x/2 \\ \mathsf{F}_R(\rho_\Delta(t,x-\Delta x),\rho_\Delta(t,x)) - \mathsf{F}_R(\rho_\Delta(t,x),\rho_\Delta(t,x+\Delta x)) & \text{if } x>\Delta x/2. \end{cases}$$

Then, at a discrete level, we have a HJ-SCL correspondence. The two schemes converge, see Guerand and Koumaiha 2019 and Andreianov and Sylla 2022.

A counter-example on a network with 3 or more branches





- There exists a unique A-**flux-limited** solution u to (HJ) on the network.
- If $N \geq 3$, the corresponding $\mathcal{G}_A$ **germ** is **not** L^1**-dissipative**.

Notables uses of the correspondence

• In Lellis, Otto, and Westdickenberg 2004; E.Y. Panov 1994, the authors prove that only one (stricty) convex entropy is required in order to caracterize the entropy solution. This result is proven in both paper by using the Hamilton-Jacobi equations theory thanks to the correspondence.

• In Colombo, Perrollaz, and Sylla 2024, the authors are able to describe a reachable set of terminal data and its corresponding initial data for heterogenous scalar conservation laws by using the HJ-SCL correspondence.

Control of discontinuous scalar conservation laws

Let H be concave and $T > 0$. Let $A : (0, T) \to [0, \max H]$. We denote by ρ^A the unique solution to

$$\begin{cases} \partial_t \rho + \partial_x H(\rho) = 0 \\ H(\rho(t,0)) \leq A(t) \\ \rho(0,x) = \rho_0(x). \end{cases}$$

We are now interested in the optimal A^* in the following minimization problem:

$$\inf_A \int_0^T \int_{\mathbb{R}} \phi(t,x)\rho^A(t,x)\,\mathrm{d}t\,\mathrm{d}x + \int_0^T g(A(t))\,\mathrm{d}t. \qquad \text{(P)}$$

Work in progress with P. Cardaliaguet and Y. Achdou

By passing to the corresponding Hamilton-Jacobi problem, we can "regularize" the control problem by studying the solution $u^{\alpha,\epsilon}$ to:

$$\partial_t u + H_\epsilon(x, \alpha(t), \partial_x u) = 0,$$

where

$$H_\epsilon(x, \alpha, p) := \begin{cases} H(p) & \text{if } |x| > \epsilon \\ \alpha H(p) & \text{if } |x| < \epsilon/2 \\ \text{a } \mathcal{C}^1 \text{ monotone regularization} & \text{else.} \end{cases}$$

Then we hope to prove that:

- we can construct an optimal control α^* to the minimization problem (P) with $u = u^{\alpha,\epsilon}$;
- we have the Γ-convergence $\alpha^* \xrightarrow[\epsilon \to 0]{} A^* / \max H$;
- A^* is an optimal in the original control problem.

On a network...

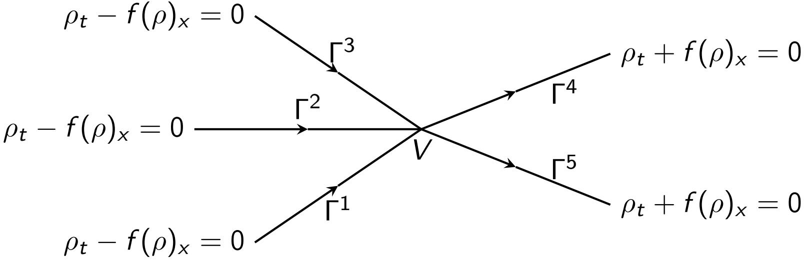


Figure: Example of a Γ_5 network.

Classifying the L^1 dissipative germs with $N \geq 3$

A computer assisted method in order to classify the different types of germs for more general junction.

$p \backslash p'$	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
S_1	O	$\sim$	O	O	$\sim$	O	$f(p_3) \leq f(p_3')$	O
S_2		$\chi_{2,3}(+)$	$\sim$	$\begin{cases} f(p_2) \geq f(p_2') \\ f(p_3) \geq f(p_3') \end{cases}$	$\begin{cases} f(p_1) \leq f(p_1') \\ f(p_2) \geq f(p_2') \end{cases}$	$f(p_2) \geq f(p_2')$	$\begin{cases} f(p_3) \leq f(p_3') \\ f(p_2) \geq f(p_2') \end{cases}$	$f(p_2) \geq f(p_2')$
S_3			O	O	$f(p_1) \leq f(p_1')$	$\sim$	$\begin{cases} f(p_3) \leq f(p_3') \\ f(p_1) \leq f(p_1') \end{cases}$	$\sim$
S_4				O	$\begin{cases} f(p_3) \leq f(p_3') \\ f(p_1) \leq f(p_1') \end{cases}$	O	$f(p_3) \leq f(p_3')$	O
S_5					$\chi_{1,3}(+)$	$\sim$	$\begin{cases} f(p_3) \leq f(p_3') \\ f(p_1) \geq f(p_1') \end{cases}$	$f(p_1) \geq f(p_1')$
S_6						O	$\begin{cases} f(p_3) \leq f(p_3') \\ f(p_2) \leq f(p_2') \end{cases}$	$\sim$
S_7							X	$\begin{cases} f(p_1) \geq f(p_1') \\ f(p_2) \geq f(p_2') \end{cases}$
S_8								$\chi_{1,2}(+)$

*W*ork in progress...

Non L^1 dissipative notion of solution ?

We can derive a notion of solution for

$$\begin{cases} \partial_t \rho_\alpha(t,x) - \partial_x \left[H_\alpha(\rho_\alpha(t,x))\right] = 0 & \text{if } \alpha \in I \\ \partial_t \rho_\alpha(t,x) + \partial_x \left[H_\alpha(\rho_\alpha(t,x))\right] = 0 & \text{if } \alpha \in J \\ F_0((\rho_\alpha(t,0^+))_{1\le\alpha\le N}) = 0, & \end{cases} \tag{3}$$

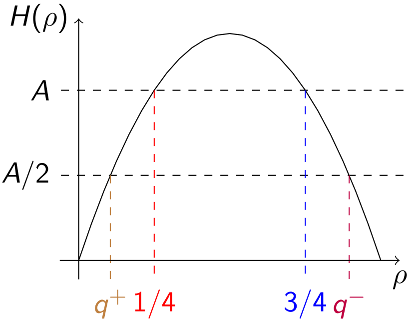
as the spatial derivative of the unique flux-limited solution of

$$\begin{cases} \partial_t u_\alpha(t,x) - H_\alpha(\partial_x u_\alpha(t,x)) = 0 & \text{if } \alpha \in I \\ \partial_t u_\alpha(t,x) - H_\alpha(\partial_x u_\alpha(t,x)) = 0 & \text{if } \alpha \in J \\ \partial_t u_\alpha(t,x) + F_0((\partial_x u_\alpha(t,0^+))_{1\le\alpha\le N}) = 0. & \end{cases} \tag{4}$$

Is this solution relevant towards applications ? Does the solution preserves the order ?

Thanks for your attention !

Simplest counter-example



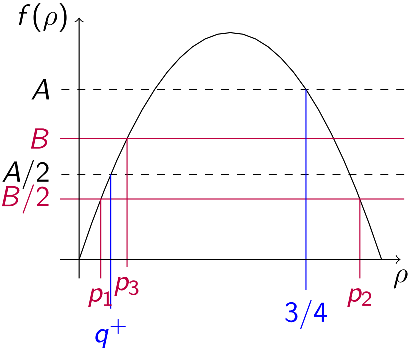
$$\mathcal{G}_1 := \{(p_1, p_2, p_3) \text{ s.t. } 2f(p_1) = 2f(p_2) = f(p_3) \text{ with } p_3 \in [3/4, 1]\}$$

$$\mathcal{G}_2 := \{(p_1, p_2, p_3) \text{ s.t. } 2f(p_1) = 2f(p_2) = f(p_3) \text{ with } p_1 \in [0, q^+]\}$$

$$\mathcal{G}_3 := \{(p_1, p_2, p_3) \text{ s.t. } 2f(p_1) = 2f(p_2) = f(p_3) \text{ with } p_2 \in [0, q^+]\}$$

$$\mathcal{G}_4 := \{(q^-, q^-, 1/4)\}$$

$$\mathcal{G}_A^{HJ} := \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4.$$



$\mathcal{G}_A^{HJ}$ is not L^1-dissipative:

$p' = (q^+, q^+, 3/4) \in \mathcal{G}_1$,

$p = (p_1, p_2, p_3) \in \mathcal{G}_2$

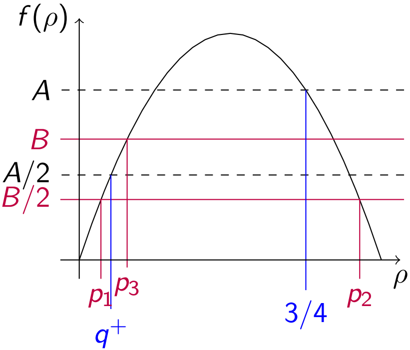
$$\mathcal{G}_1 := \{(p_1, p_2, p_3) \text{ s.t. } 2f(p_1) = 2f(p_2) = f(p_3) \text{ with } p_3 \in [3/4, 1]\}$$

$$\mathcal{G}_2 := \{(p_1, p_2, p_3) \text{ s.t. } 2f(p_1) = 2f(p_2) = f(p_3) \text{ with } p_1 \in [0, q^+]\}$$

$$\mathcal{G}_3 := \{(p_1, p_2, p_3) \text{ s.t. } 2f(p_1) = 2f(p_2) = f(p_3) \text{ with } p_2 \in [0, q^+]\}$$

$$\mathcal{G}_4 := \{(q^-, q^-, 1/4)\}$$

$$\mathcal{G}_A^{HJ} := \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4.$$



$\mathcal{G}_A^{HJ}$ is not L^1-dissipative:

$p' = (q^+, q^+, 3/4) \in \mathcal{G}_1$,

$p = (p_1, p_2, p_3) \in \mathcal{G}_2$

$$\begin{aligned}\operatorname{sign}(p_1 - p_1')\left[f(p_1) - f(p_1')\right] + \operatorname{sign}(p_2 - p_2')\left[f(p_2) - f(p_2')\right]\\ - \operatorname{sign}(p_3 - p_3')\left[f(p_3) - f(p_3')\right]\\ = (B/2 - A/2) - (B/2 - A/2) + (B - A) = B - A < 0.\end{aligned}$$

Ideas on the computer assisted method

$p \backslash p'$	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
S_1	O	$\sim$	O	O	$\sim$	O	$f(p_3) \leq f(p_3')$	O
S_2		$\chi_{2,3}(+)$	$\sim$	$\begin{cases} f(p_2) \geq f(p_2') \\ f(p_3) \geq f(p_3') \end{cases}$	$\begin{cases} f(p_1) \leq f(p_1') \\ f(p_2) \geq f(p_2') \end{cases}$	$f(p_2) \geq f(p_2')$	$\begin{cases} f(p_3) \leq f(p_3') \\ f(p_2) \geq f(p_2') \end{cases}$	$f(p_2) \geq f(p_2')$
S_3			O	O	$f(p_1) \leq f(p_1')$	$\sim$	$\begin{cases} f(p_3) \leq f(p_3') \\ f(p_1) \leq f(p_1') \end{cases}$	$\sim$
S_4				O	$\begin{cases} f(p_3) \leq f(p_3') \\ f(p_1) \leq f(p_1') \end{cases}$	O	$f(p_3) \leq f(p_3')$	O
S_5					$\chi_{1,3}(+)$	$\sim$	$\begin{cases} f(p_3) \leq f(p_3') \\ f(p_1) \geq f(p_1') \end{cases}$	$f(p_1) \geq f(p_1')$
S_6						O	$\begin{cases} f(p_3) \leq f(p_3') \\ f(p_2) \leq f(p_2') \end{cases}$	$\sim$
S_7							X	$\begin{cases} f(p_1) \geq f(p_1') \\ f(p_2) \geq f(p_2') \end{cases}$
S_8								$\chi_{1,2}(+)$