A numerical scheme for the discontinuous Eikonal equation

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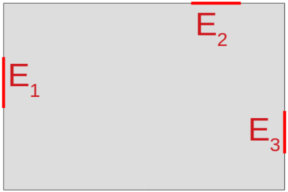
We want to model a moving crowd. The crowd is represented as a pedestrian density $\rho(t, x)$ between 0 and 1.
Starting at $t = 0$, the pedestrians want to move out of the room using the exit(s).



$E = E_1 \cup E_2 \cup E_3$

In the 1D setting, we model the transport using a scalar conservation law:

$$\rho_t + f(\rho)_x = 0.$$

The flux is equal to the density multiply by the speed of agents.

$$f(s,x) := \rho(s,x)v(s,x)$$

The velocity v is itself governed by the local density:

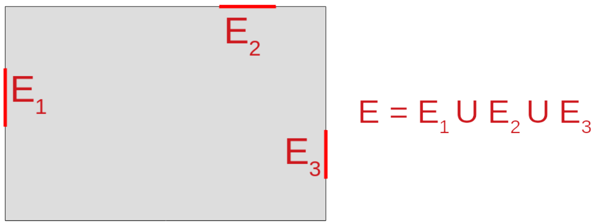
$$v(s,x) := v_{\max}(1-\rho)$$

We set $v_{\max} = 1$ and recover:

$$f(s,x) := f(\rho(s,x)) := \rho(s,x)(1-\rho(s,x))$$

• M. J. Lighthill and G. B. Whitham, On kinematic waves. ii. a theory of traffic flow on long crowded roads, (1955).

Back to the initial problem, at $t = 0$, the agents want to exit the room minimizing their exit time (or total cost...).



Suppose $V(t,x) \in \mathcal{S}^1$ is a vector field corresponding to the choice of direction of an agent located in x at time t. Then the density equation follows from LWR:

$$\rho_t + \mathrm{div}_x(V(t,x)\rho v(\rho)) = 0.$$

How do we compute V ?

For a fixed density $\rho(x)$, we use an optimal control problem.
Fix a density ρ in a given domain Ω. Let $\alpha(\cdot) \in \mathcal{C}^1([0,+\infty), \mathcal{S}^1)$. Consider the following dynamic for the controlled trajectories y_x solution of the Cauchy problem:

$$\begin{cases} \dot{y}_x(t) = v(\rho(y_x(t)))\alpha(t) \\ y_x(0) = x. \end{cases}$$

In order to model the "disconfort" one can experiment by staying in high density regions, we use a running cost function $g(\rho)$ increasing with respect to the density. Also, since each agent seeks to minimize its exit cost, we assume $g > 0$. We define the value function:

$$u(x) = \inf_{\alpha(\cdot)} \int_0^\infty g(\rho(y_x(t)))\mathbb{1}_\Omega(y_x(t))\,\mathrm{d}t.$$

Heuristically, suppose that the infinum is a minimum reached for an optimal trajectory $y_x^\star(\cdot)$.

The pedestrian at x should follow the direction field $V(x) = \dot{y}_x^\star(0)$.

Then, using the dynamic programming principle, we should have

$$V(x) = \dot{y}_x^\star(0) = -\frac{\nabla u(x)}{||\nabla u(x)||}.$$

For a fixed ρ, using the classical Hamilton-Jacobi-Bellman approach, we want to find the gradient of the viscosity solution the following eikonal equation:

$$||\nabla u|| = \frac{g(\rho)}{v(\rho)} =: c(x).$$

Two big criticism of this model :

• For any t, each agent instantaneously knows the density of the crowd in the whole domain.

• The agents do not anticipate the movement the other pedestrian.

To summarize, we should find the solutions of the Hughes model:

$$\begin{cases} \rho_t + \mathrm{div}_x\left(\frac{-\nabla u}{|\nabla u|}\rho v(\rho)\right) = 0 \\ |\nabla_x u| = \frac{g(\rho)}{v(\rho)} \\ u(x \in E) = 0 \\ \rho(0, x) = \rho_0(x) \end{cases} \tag{1}$$

where g is a given cost function depending on the local density.

For the one-dimensional problem, there exist a few existence results:

- B. Andreianov, M. Rosini, and G. Stivaletta. On existence, stability and many-particle approx- imation of solutions of 1D Hughes model with linear costs, 2021.
- B. Andreianov, T. Girard, Existence of solutions for a class of one-dimensional models of pedestrian evacuations, SIAM J. Math. Anal. 56 (3), 2024.
- Halvard Olsen Storbugt. Convergence of rough follow-the-leader approximations and existence of weak solutions for the one-dimensional Hughes model. Discrete and Continuous Dynamical Systems, 2024.

The 2D case is still an open problem...

In the following we focus mainly on the problem:

$$\begin{cases} ||\nabla u|| = c(x) & \text{in } \Omega \\ u = 0 & \text{in } E, \end{cases} \tag{2}$$

where

$$c \in L^\infty(\Omega, (\underline{c}, \bar{c})),$$
$$0 < \underline{c} < \bar{c}.$$

The classical viscosity solutions for the eikonal equation:

Definition (Subsolution (resp. supersolution) to (2))

We say that $u \in \mathcal{C}^0(\bar{\Omega})$ is a subsolution (resp. supersolution) to (2) if, for any $\psi \in \mathcal{C}^1(\bar{\Omega})$, if $u - \psi$ admits a maximum (resp. a minimum) at x_0, we have

$$||\nabla\psi(x_0)|| \leq \text{(resp. } \geq \text{)}\ c(x_0) \text{ if } x_0 \in \Omega$$
$$u(x_0) = 0 \text{ if } x_0 \in E$$

Problem : The uniqueness proof relies on the continuity of the source term c.

Let $(U_i)_i$ be a family of open sets such that $\mathrm{meas}\big(\Omega - \bigcup_i U_i\big) = 0$ and v is continuous on each U_i. We can define the solution to the scalar conservation law:

Definition

Let $\rho_0 \in L^\infty(\Omega)$. We say that $\rho \in L^\infty$ is a solution to (1) iif ρ is a weak solution and for any $i \in I$, for any non-negative $\phi \in \mathcal{C}_c^\infty([0, T) \times U_i)$, for any k,

$$\iint_{(0,T)\times U_i} |\rho - k|\phi_t + \mathrm{sign}(\rho - k)\,[f(\rho) - f(k)]\, v(x) \cdot \nabla_x(\phi)\,\mathrm{d}t\,\mathrm{d}x$$

$$- \iint_{(0,T)\times U_i} \mathrm{sign}(\rho - k) f(k)\,\mathrm{div}(v(x))\phi\,\mathrm{d}t\,\mathrm{d}x$$

$$+ \int_{U_i} |\rho_0 - k|\,\phi(0,x)\,\mathrm{d}x \geq 0$$

Discontinuous viscosity solutions:

- The stratified approach
- flux-limited solutions (Imbert, Monneau)
- junction viscosity solutions (Lions, Souganidis)

General reference : G. Barles and E. Chasseigne. On Modern Approaches of Hamilton-Jacobi Equations and Control Problems with Discontinuities. Springer, 2024.

Drawback : one needs to know precisely where the discontinuities are beforehand in order to use these notions of solution.

Monge solutions :
R. Newcomb and Jianzhong Su. Eikonal equations with discontinuities. Differential and Integral Equations, 1995.

Let $T > 0$, we denote by Γ_x the set :

$$\Gamma_x := \{\gamma \in W^{1,1}([0,T],\bar{\Omega}) \text{ s.t. } \forall t \in [0,T], \gamma(t) \in \bar{\Omega}, \gamma(0) = x\}.$$

We define :

$$L(x,y) = \inf_{\substack{\gamma \in \Gamma_x \\ \gamma(T) = y}} \int_0^T c(\gamma(t))|\dot{\gamma}(t)|\,\mathrm{d}t. \tag{3}$$

Definition

We say that u is a Monge subsolution (resp. supersolution) with state-constraint to

$$\begin{cases} \|\nabla u(x)\| = c(x) & \text{if } x \in \bar{\Omega} - E \\ u(x) = 0 & \text{if } x \in E \end{cases}$$

if for any $x_0 \in \bar{\Omega} - E$

$$\lim_{x \to x_0} \inf \frac{u(x) - (u(x_0) - L(x_0, x))}{|x - x_0|} \geq 0 (\text{ resp. } \leq 0) \tag{4}$$

and $\forall x \in E, u(x) = 0$.

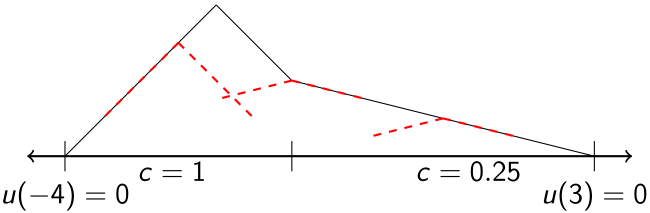
If c is l.s.c. then u is unique.

Heuristics.

Lemma (A Monge solution lies on its lower Monge cones)

If $u \in \mathcal{C}^0(\bar{\Omega}) \bigcap W^{1,\infty}(\Omega)$ is a Monge subsolution of (4) then for any $x_0 \in \bar{\Omega}$ there exists $r > 0$ such that for any $x \in B_r(x_0) \bigcap \bar{\Omega}$,

$$u(x) \geq u(x_0) - L(x, x_0). \tag{5}$$



Maximal Lipschitz viscosity subsolution VS Monge solution

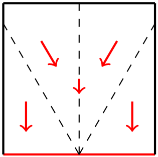
$$\Omega = [-1,1] \times [0,2], \quad E = [-1,1] \times \{0\}$$

$$c(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2 & \text{else} \end{cases}$$

Maximal Lipschitz subsolution

$$u_1(x, y) = 2y$$

Monge solution

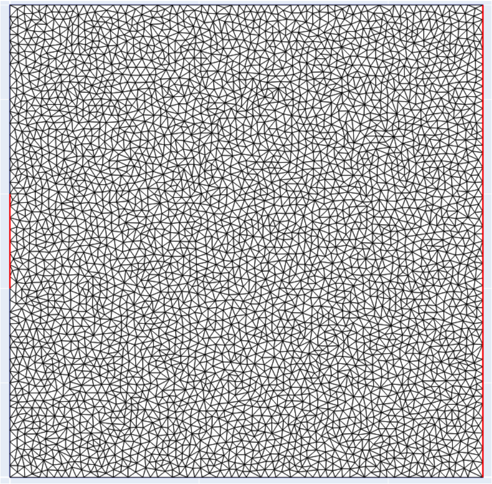


Closest existing result:
A. Festa and M. Falcone. "An Approximation Scheme for an Eikonal Equation with Discontinuous Coefficient", SIAM J. Numer. Anal., 52(1) (2014): 236-257
Convergence of a numerical scheme (following the semi-lagrangian approach) under the cone assumption on c :
$\forall \eta > 0, K > 0, \forall x \in \Omega, \exists n_x \in \mathcal{S}^1, \forall y \in B(x, \eta), \forall r > 0, \forall d \in \mathcal{S}^1$ s.t. $|d - n_x| < \eta, y + rd \in \Omega,$

$$c(y + rd) - c(y) \leq Kr.$$

Under this assumption, the Ishii solution u is unique and the numerical scheme converges to u. However, in the previous example, both the Lipschitz and the Monge solution are Ishii solutions.

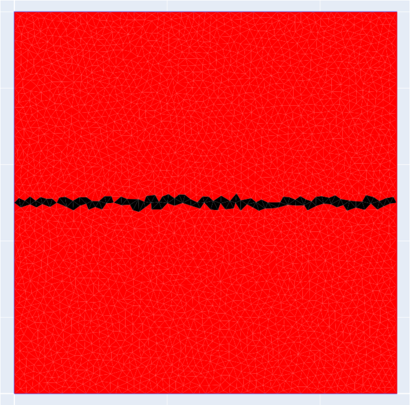
We want to approximate the Monge solution of the Eikonal equation on a triangular mesh $M_\Delta := (\mathcal{T}_n)_{1 \leq n \leq N}$.



We discretize the source term:

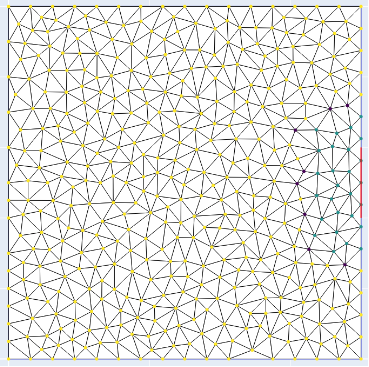
$$\forall n \in [\![1, N]\!], c_n := \inf_{x \in \mathcal{T}_n} c(x), \tag{6}$$

$$c_\Delta(x) := \sum_n \mathbb{1}_{\mathcal{T}_n}(x) c_n.$$

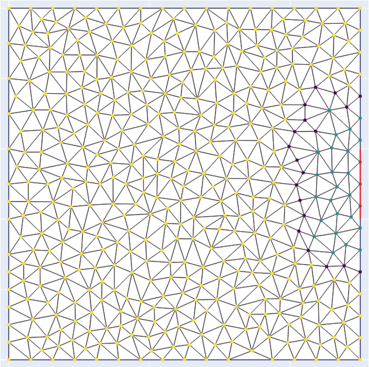


The fast marching principle and the narrow band depth

Narrow band depth 1



Narrow band depth 2



The computations inside a triangle.
In the following, we denote $\mathcal{T}_n = ABC$ and $V_{A,B,C}$ stands for $u_\Delta(A,B,C)$. We define:

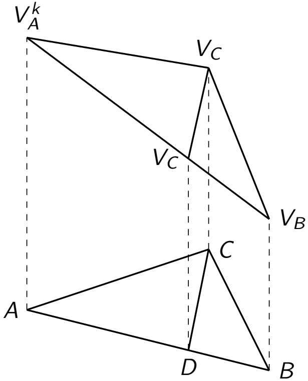
$$\Phi_{ABC}^{V_A,V_B,V_C} : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R} \\ x\overrightarrow{AB} + y\overrightarrow{AC} \mapsto V_A + (V_B - V_A)x + (V_C - V_A)y. \end{cases} \tag{7}$$

Then the gradient of $\Phi_{ABC}^{V_A,V_B,V_C}$ is constant on $\mathbb{R}^2$. Consequently, we define:

$$\mathcal{H}_{ABC}(V_A, V_B, V_C) = ||\nabla\Phi_{ABC}^{V_A,V_B,V_C}||.$$

For any triangle $\mathcal{T}_k$ with two validated points B and C with $V_B < V_C$, we set $V_A^k =$

$$\begin{cases} V_B - \frac{\vec{AB}\cdot\vec{BC}(V_C - V_B)}{BC^2} + \frac{|\det(\vec{AB},\vec{BC})|\sqrt{c_\Delta^2 BC^2 - (V_B - V_C)^2}}{BC^2} & \text{if } c_\Delta |BC| > |V_B - V_C| \\ V_C + c_\Delta\, AC & \text{else} \end{cases}$$



If instead we compute :

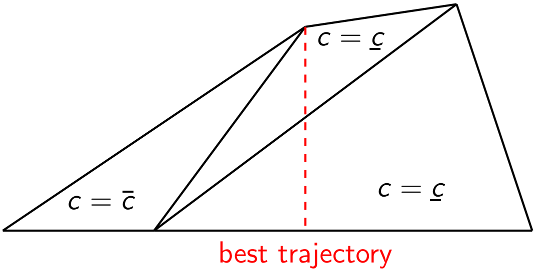
$$\tilde{V}_A^k := \inf_{\substack{\gamma \in W^{1,\infty}((0,T)) \\ \gamma(0)=A \\ \gamma(T)\in[BC]}} \int_0^T c_\Delta |\dot{\gamma}(t)| \,\mathrm{d}t + U_{BC}(\gamma(T)).$$

We obtain $\tilde{V}_A^k =$

$$\begin{cases} V_B + c_\Delta |AB| & \text{if } c_\Delta \frac{\vec{AB}\cdot\vec{BC}}{AB} + V_C - V_B > 0 \\ V_C + c_\Delta |AC| & \text{if } c_\Delta \frac{\vec{AC}\cdot\vec{BC}}{AC} + V_C - V_B < 0 \\ V_B - \frac{\vec{AB}\cdot\vec{BC}(V_C - V_B)}{BC^2} + \frac{|\det(\vec{AB},\vec{BC})|\sqrt{c_\Delta^2 BC^2 - (V_B - V_C)^2}}{BC^2} & \text{else} \end{cases}$$

A choice: do we use the constrained gradient or not ?

The issue with obtuse triangles



Let P be the point phrozen at the step n. We set

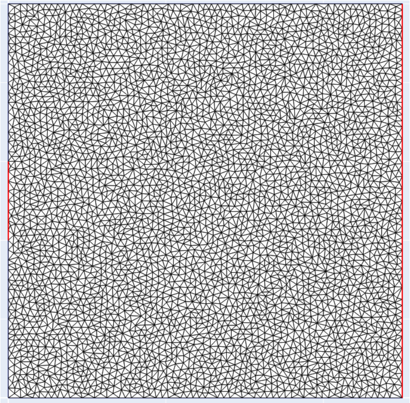
$$u_\Delta(P) = \min_{P\in\mathcal{T}_k} V_A^k.$$

Good properties

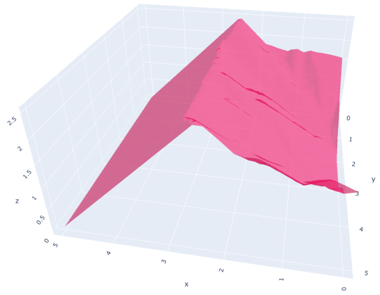
- (*Monotonicity*) If $c_\Delta \leq \tilde{c}_\Delta$ then $u_\Delta \leq \tilde{u}_\Delta$.
- (*Compactness*) $||\nabla u_\Delta|| \leq \bar{c}$.
- (*Partial consistency*) Let $\phi \in \mathcal{C}^1(\Omega)$. Then

$$\limsup_{y\to x,\ y\in ABC} [\mathcal{H}_{ABC}(\phi(A),\phi(B),\phi(C)) - c_\Delta(y)] \leq ||\nabla\phi(x)|| - c(x). \tag{8}$$

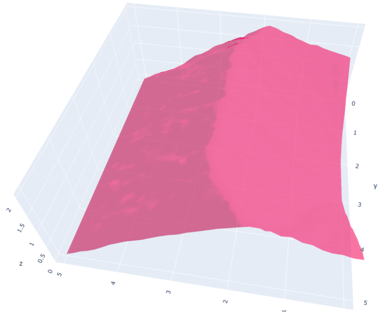
Comparison of the numerical approximations with different options



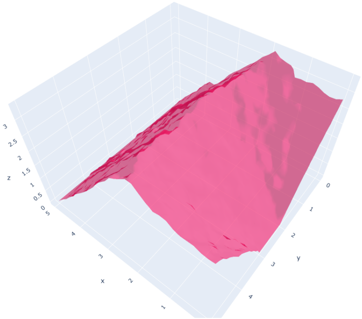
Narrow band depth 1, unconstrained gradient



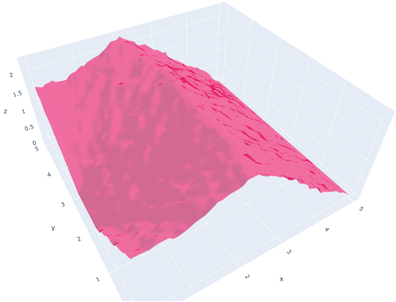
Narrow band depth 2, unconstrained gradient



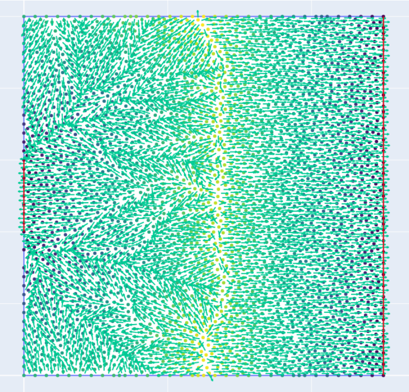
Narrow band depth 1, constrained gradient



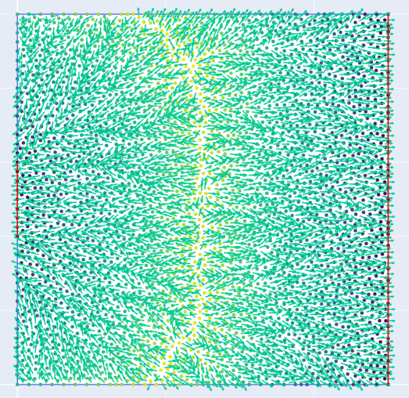
Narrow band depth 2, constrained gradient



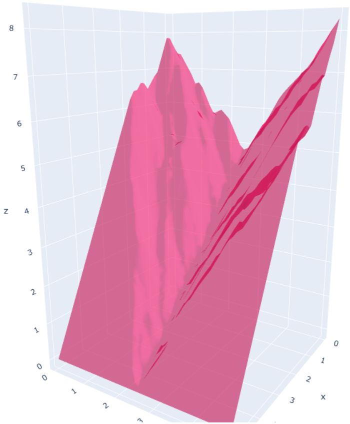
Narrow band depth 1, unconstrained gradient



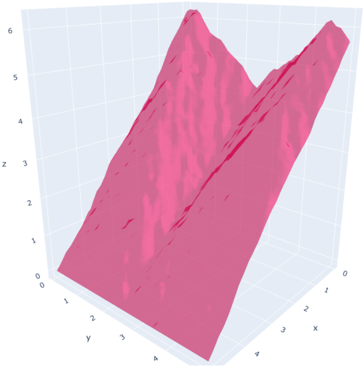
Narrow band depth 2, unconstrained gradient



Narrow band depth 1, unconstrained gradient



Narrow band depth 2, constrained gradient



Hughes 2D in the university restaurant of Tours

Thanks for your attention.